

since

$$\frac{d}{dx} L_\nu^{(\alpha)}(x) = -L_{\nu-1}^{(\alpha+1)}(x). \tag{4.6}$$

Then, we can easily deduce that the polynomials $\{q_\nu^{(\alpha)}\}_{\nu \geq 0}$, defined by

$$q_\nu^{(\alpha)}(x) = \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \sum_{j=0}^{\nu} (-1)^j \binom{\nu+\alpha}{\nu-j} \frac{x^j}{j!}, \quad \nu \geq 0,$$

are orthonormal with respect to the Sobolev inner product (4.5).

On the other hand, using (4.6) we can deduce that

$$\frac{d^r}{dx^r} L_\nu^{(\alpha)}(x) = (-1)^r L_{\nu-r}^{(\alpha+r)}(x), \quad 1 \leq r \leq \nu.$$

Hence, for $1 \leq r \leq \nu$, we obtain

$$\frac{d^r}{dx^r} q_\nu^{(\alpha)}(x) = (-1)^r \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} L_{\nu-r}^{(r+\alpha)}(x). \tag{4.7}$$

Now, using the addition formula (see [29, p. 391, Problem 90])

$$L_\nu^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{\nu} L_{\nu-k}^{(\alpha)}(x) L_k^{(\beta)}(y),$$

we can deduce that equality (4.7) is equivalent to:

$$\begin{aligned} \frac{d^r}{dx^r} q_\nu^{(\alpha)}(x) &= (-1)^r \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \sum_{k=0}^{\nu-r} \binom{r+k-1}{k} L_{\nu-r-k}^{(\alpha)}(x) \\ &= (-1)^r \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \sum_{k=0}^{\nu-r} \tilde{e}_{k,\nu-r} q_{\nu-r-k}^{(\alpha)}(x) \end{aligned} \tag{4.8}$$

where

$$\tilde{e}_{k,\nu-r} = \left(\frac{(\nu-r-k+1)\Gamma(\nu-r-k+\alpha+1)}{(\nu-r-k)!} \right)^{\frac{1}{2}} \binom{r+k-1}{k}.$$

Substituting $j = \nu - r - k$ in (4.8), we have

$$\frac{d^r}{dx^r} q_\nu^{(\alpha)}(x) = (-1)^r \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \sum_{j=0}^{\nu-r} \tilde{e}_{\nu-r-j,\nu-r} q_j^{(\alpha)}(x).$$

Again, if we take $T = \frac{d^r}{dx^r}$, then the relation (4.8) implies that the coefficients in (3.2) are given by

$$d_\nu^j = \begin{cases} (-1)^r \left(\frac{\nu!(j+1)\Gamma(j+\alpha+1)}{j!(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \binom{\nu-j-1}{\nu-j-r}, & 0 \leq j \leq \nu-r, \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$n(T) = \sup_{\substack{\|c\|_2=1 \\ c \in \mathbb{C}^{n+1}}} \left\{ \sum_{j=0}^{n-r} \left| \sum_{\nu=0}^n c_\nu d_\nu^j \right|^2 \right\}^{\frac{1}{2}}.$$

Furthermore,

$$\begin{aligned} \max_{r \leq \nu \leq n} \left\| \frac{d^r}{dx^r} q_\nu^{(\alpha)} \right\|_S &= \max_{r \leq \nu \leq n} \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \left\| \sum_{j=0}^{\nu-r} \tilde{\epsilon}_{\nu-r-j, \nu-r} q_j^{(\alpha)} \right\|_S \\ &= \max_{r \leq \nu \leq n} \left(\frac{\nu!}{(\nu+1)\Gamma(\nu+\alpha+1)} \right)^{1/2} \Delta_\nu^{(\alpha)}, \end{aligned}$$

where

$$\Delta_\nu^{(\alpha)} = \left(\sum_{j=0}^{\nu-r} \frac{(j+1)\Gamma(j+\alpha+1)}{j!} \left(\frac{\nu-j-1}{\nu-j-r} \right)^2 \right)^{1/2}, \quad r \leq \nu \leq n.$$

Since the binomial coefficient is an increasing function of ν , we have

$$\max_{r \leq \nu \leq n} \Delta_\nu^{(\alpha)} = \Delta_n^{(\alpha)}.$$

Also, $\frac{\nu!}{\Gamma(\nu+\alpha+1)}$ is decreasing (increasing) if $\alpha \geq 0$ ($-1 < \alpha < 0$), (see [4] or [13]), then

$$\begin{aligned} a_{n,n}^{(\alpha)} \Delta_n^{(\alpha)} &\leq \max_{r \leq \nu \leq n} \left\| \frac{d^r}{dx^r} q_\nu^{(\alpha)} \right\|_S \leq a_{r,r}^{(\alpha)} \Delta_n^{(\alpha)}, \quad \text{if } \alpha \geq 0, \\ a_{r,n}^{(\alpha)} \Delta_n^{(\alpha)} &\leq \max_{r \leq \nu \leq n} \left\| \frac{d^r}{dx^r} q_\nu^{(\alpha)} \right\|_S \leq a_{n,r}^{(\alpha)} \Delta_n^{(\alpha)}, \quad \text{if } -1 < \alpha < 0, \end{aligned}$$

where

$$a_{n,r}^{(\alpha)} = \left(\frac{n!}{(r+1)\Gamma(n+\alpha+1)} \right)^{1/2}.$$

Hence, inequality (3.4) gives

$$\begin{cases} a_{n,n}^{(\alpha)} \Delta_n^{(\alpha)} \leq n(T) \leq (n+1)^{1/2} a_{r,r}^{(\alpha)} \Delta_n^{(\alpha)}, & \text{if } \alpha \geq 0, \\ a_{r,n}^{(\alpha)} \Delta_n^{(\alpha)} \leq n(T) \leq (n+1)^{1/2} a_{n,r}^{(\alpha)} \Delta_n^{(\alpha)}, & \text{if } -1 < \alpha < 0. \end{cases}$$

In particular, when $\alpha = 0$, we obtain

$$\left(\sum_{j=0}^{n-r} \frac{(j+1)}{(n+1)} \left(\frac{n-j-1}{n-j-r} \right)^2 \right)^{1/2} \leq n(T) \leq \left(\sum_{j=0}^{n-r} \frac{(n+1)(j+1)}{r+1} \left(\frac{n-j-1}{n-j-r} \right)^2 \right)^{1/2}.$$

Finally, other examples arise when the vector of measures (μ_0, μ_1) is either a coherent or a symmetrically coherent pair (see[12], [19]), and the Sobolev inner

product on $\mathbb{P}(\mathbb{C})$ is defined as in (2.1). Meijer has proved that if (μ_0, μ_1) is either a coherent or a symmetrically coherent pair, then at least μ_0 or μ_1 is a classical measure (that is, Jacobi, Laguerre or Hermite measure), and has completely classified all the coherent and the symmetrically coherent pairs (see [19]). In these cases, applying Theorem 3.1 with an appropriate basis $\{\phi_\nu\}_{\nu \geq 0}$, it is possible to determine an expression for ${}_n(T)$ when $T = \frac{d^r}{dx^r}$, $1 \leq r \leq n$.

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ON THE CONVERGENCE AND ADHERENCE OF GRILLS

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ABSTRACT. We define the convergence and adherence of grill in topological space with respect to the family of all α -open sets, semiopen sets, preopen sets, b -open sets and β -open sets, characterize such sets and discuss their properties.

1. INTRODUCTION AND PRELIMINARIES

Grills in a topological space (X, τ) is initiated by G. Choquet [6] in 1947. Properties of grill in topological spaces are further investigated in [8, 10, 11, 15, 16]. In this paper, we define the convergence and adherence of grill in the following family of sets obtained from τ , namely the family of all semiopen sets (σ), preopen sets (π), b -open sets (b) and β -open sets (β).

A grill \mathcal{G} [6] is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{G}$ and $A \subset B \subset X$ implies $B \in \mathcal{G}$ and (ii) $A, B \subset X$ and $A \cup B \in \mathcal{G}$ implies $A \in \mathcal{G}$ or $B \in \mathcal{G}$. By a space (X, τ) , we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\text{cl}(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of A in (X, τ) . A subset A of a topological space is said to be semiopen [9] (resp. preopen [1], b -open [4], β -open [2] or semipreopen [3]) if $A \subset \text{cl}(\text{int}(A))$ (resp. $A \subset \text{int}(\text{cl}(A))$, $A \subset \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$, $A \subset \text{cl}(\text{int}(\text{cl}(A)))$.) A filter \mathcal{F} on a space X is said to θ -adheres at $x \in X$ if for each $F \in \mathcal{F}$ and each open set U containing x , $F \cap \text{cl}(U) \neq \emptyset$. \mathcal{F} is said to θ -converge to $x \in X$ if for each open set U containing x , there corresponds $F \in \mathcal{F}$ such that $F \subset \text{cl}(U)$. If open sets are replaced by preopen sets, we get $p(\theta)$ -convergence and $p(\theta)$ -adherence defined and discussed in [10]. If \mathcal{G} is a grill (or filter) on a space X , then the *section of \mathcal{G}* , [10] denoted by $\text{sec } \mathcal{G}$, is given by $\text{sec } \mathcal{G} = \{A \subset X \mid A \cap G \neq \emptyset, \text{ for all } G \in \mathcal{G}\}$. Let (X, τ) be a space and $\mathcal{P} = \{\sigma, \pi, b, \beta\}$. For $\mu \in \mathcal{P}$ and a subset A of X , $i_\mu(A)$ is the union of all μ -open subsets of A and $c_\mu(A)$ is the intersection of the μ -closed supersets of A . Clearly, for $x \in X$ and $A \subset X$, $x \in c_\mu(A)$ if and only if $U \cap A \neq \emptyset$ for all $U \in \mu$ containing x . For any subset A of X , let us denote by $c_{\theta_\mu}(A)$, the

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set of all points $x \in X$ such that $c_\mu(U) \cap A \neq \emptyset$ whenever $x \in U \in \mu$. That is, $c_{\theta_\mu}(A) = \{x \in X \mid A \cap c_\mu(U) \neq \emptyset \text{ for every } \mu\text{-open set } U \text{ containing } x\}$. The following lemma will be useful in the sequel.

Lemma 1.1. [16] *Let (X, τ) be a space. Then the following hold.*

- (a) *For any grill (resp. filter) \mathcal{G} on a space X , $\text{sec } \mathcal{G}$ is a filter (resp. grill) on X .*
- (b) *If \mathcal{F} is a filter and \mathcal{G} is a grill such that $\mathcal{F} \subset \mathcal{G}$, then there is an ultrafilter \mathcal{U} on X such that $\mathcal{F} \subset \mathcal{U} \subset \mathcal{G}$.*

2. MAIN THEOREMS

Let (X, τ) be a space and $\mathcal{P} = \{\sigma, \pi, b, \beta\}$. Throughout the paper, $\mu \in \mathcal{P}$. A grill \mathcal{G} on X is said to be θ_μ -adheres at x in X if for each $U \in \mu$ containing x and each $G \in \mathcal{G}$, $c_\mu(U) \cap G \neq \emptyset$. \mathcal{G} is said to θ_μ -converges to $x \in X$ if for each $U \in \mu$ containing x , there exists $G \in \mathcal{G}$ such that $G \subset c_\mu(U)$. Clearly, a grill \mathcal{G} is θ_μ -converges to $x \in X$ if and only if \mathcal{G} contains the collection $\{c_\mu(A) \mid x \in A \in \mu\}$. A filter \mathcal{F} on a topological space (X, τ) is said to be θ_μ -adheres at $x \in X$ if for each $U \in \mu$ containing x and each $F \in \mathcal{F}$, $c_\mu(U) \cap F \neq \emptyset$. \mathcal{F} is said to θ_μ -converges to $x \in X$ if for each $U \in \mu$ containing x , there exists $F \in \mathcal{F}$ such that $F \subset c_\mu(U)$. The following Theorem 2.1 shows that θ_μ -adherence of a grill is strictly stronger than that of θ_μ -convergence.

Theorem 2.1. *Let (X, τ) be a space with a grill \mathcal{G} . If \mathcal{G} θ_μ -adheres at some point $x \in X$, then \mathcal{G} is θ_μ -convergent to x .*

Proof. Suppose \mathcal{G} θ_μ -adheres at $x \in X$. Then for each μ -open set U containing x and each $G \in \mathcal{G}$, $c_\mu(U) \cap G \neq \emptyset$. Now $c_\mu(U) \cap G \neq \emptyset$ implies that $c_\mu(U) \in \text{sec } \mathcal{G}$ for each μ -open set U containing x . Moreover, $X - c_\mu(U) \notin \mathcal{G}$ whenever $x \in U \in \mu$. Since $X \in \mathcal{G}$ and \mathcal{G} is a grill on X , $c_\mu(U) \in \mathcal{G}$ for each μ -open set containing x . Thus, \mathcal{G} is θ_μ -convergent to x . \square

If $\mu = \pi$ in the above Theorem 2.1, we get Theorem 2.6 of [10] and Example 2.7 of [10] shows that the converse of the above theorem is not true for $\mu = \pi$. Let (X, τ) be a space, \mathcal{G} be a grill on X and $x \in X$. We define $\mathcal{G}(\theta_\mu, x) = \{A \subset X \mid x \in c_{\theta_\mu}(A)\}$ and $\text{sec } \mathcal{G}(\theta_\mu, x) = \{A \subset X \mid A \cap G \neq \emptyset \text{ for all } G \in \mathcal{G}(\theta_\mu, x)\}$. The following Theorem 2.2 gives a characterization of θ_μ -adherence of a grill in terms of $\mathcal{G}(\theta_\mu, x)$ which is nothing but Theorem 2.9 of [10] if $\mu = \pi$ and Theorem 2.3 gives a characterization of θ_μ -convergence of a grill which is nothing but Theorem 2.10 of [10] if $\mu = \pi$.

Theorem 2.2. *Let \mathcal{G} be a grill on a space (X, τ) . Then the grill \mathcal{G} θ_μ -adheres at $x \in X$ if and only if $\mathcal{G} \subset \mathcal{G}(\theta_\mu, x)$.*

Proof. \mathcal{G} θ_μ -adheres at $x \in X$ implies that $c_\mu(U) \cap G \neq \emptyset$ for all $G \in \mathcal{G}$ and for every μ -open set U containing x . Therefore, $x \in c_{\theta_\mu}(G)$ for all $G \in \mathcal{G}$ and so $G \in \mathcal{G}(\theta_\mu, x)$ for all $G \in \mathcal{G}$ which implies that $\mathcal{G} \subset \mathcal{G}(\theta_\mu, x)$. Conversely, suppose that $\mathcal{G} \subset \mathcal{G}(\theta_\mu, x)$. Then for every $G \in \mathcal{G}$, $G \in \mathcal{G}(\theta_\mu, x)$ and so $x \in c_{\theta_\mu}(G)$ which implies that $c_\mu(U) \cap G \neq \emptyset$ for every $U \in \mu$ containing x . Hence \mathcal{G} θ_μ -adheres at the point x . □

Theorem 2.3. *Let \mathcal{G} be a grill on a space (X, τ) . Then \mathcal{G} is θ_μ -convergent to $x \in X$ if and only if $\text{sec } \mathcal{G}(\theta_\mu, x) \subset \mathcal{G}$.*

Proof. Let \mathcal{G} be a grill on X which θ_μ -converges to $x \in X$. Then for each μ -open set U containing x , there exists a $G \in \mathcal{G}$ such that $G \subset c_\mu(U)$. Since \mathcal{G} is a grill on X , $c_\mu(U) \in \mathcal{G}$. Let $A \in \text{sec } \mathcal{G}(\theta_\mu, x)$. Then $A \cap G \neq \emptyset$ for all $G \in \mathcal{G}(\theta_\mu, x)$ and so $X - A \notin \mathcal{G}(\theta_\mu, x)$. Now $X - A \notin \mathcal{G}(\theta_\mu, x)$ implies that $x \notin c_{\theta_\mu}(X - A)$ and so $c_\mu(V) \cap (X - A) = \emptyset$ for some $V \in \mu$ containing x . Again, $c_\mu(V) \cap (X - A) = \emptyset$ implies that $c_\mu(V) \subset A$ which in turn implies that $A \in \mathcal{G}$. Thus, $\text{sec } (\theta_\mu, x) \subset \mathcal{G}$. Conversely, suppose that \mathcal{G} does not θ_μ -converge to $x \in X$. Then there exists a μ -open set U containing x such that $c_\mu(U) \notin \mathcal{G}$ and so $c_\mu(U) \notin \text{sec } \mathcal{G}(\theta_\mu, x)$. Since $c_\mu(U) \notin \text{sec } \mathcal{G}(\theta_\mu, x)$, there exists $G \in \mathcal{G}(\theta_\mu, x)$ such that $c_\mu(U) \cap G = \emptyset$. Since $G \in \mathcal{G}(\theta_\mu, x)$, $x \in c_{\theta_\mu}(G)$ and so $c_\mu(U) \cap G \neq \emptyset$ for all μ -open set U containing x , a contradiction. Hence \mathcal{G} θ_μ -converges to x . □

A nonempty set A of a space (X, τ) is said to be μ -closed relative to X if for every cover $\{V_\alpha \mid \alpha \in \Delta\}$ of A by μ -open sets of X , there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{c_\mu(V_\alpha) \mid \alpha \in \Delta_0\}$. A space X is μ -closed if $A = X$. If $\mu = \tau$, then the family of all τ -closed spaces coincides with the family of all almost compact(QHC) spaces [14], if $\mu = \pi$, then the family of all π -closed spaces coincides with the family of all p -closed spaces [1, 10], if $\mu = \beta$, then we have the family of all β -closed spaces [5], if $\mu = b$, then we have the family of all b -closed spaces [13] and if $\mu = \sigma$, then the family of all σ -closed spaces coincides with the family of all s -closed spaces [7]. The following Theorem 2.4 characterizes μ -closed spaces, which is also true for $\mu = \tau$.

Theorem 2.4. *Let (X, τ) be a space and $\mu \in \mathcal{P}$. Then the following are equivalent.*

- (a) X is μ -closed.
- (b) Every maximal filterbase θ_μ -converges to some point of X .
- (c) Every filterbase θ_μ -adheres at some point of X .
- (d) For every family $\{V_\alpha \mid \alpha \in \Delta\}$ of μ -closed subsets such that $\cap\{V_\alpha \mid \alpha \in \Delta\} = \emptyset$, there exists a finite subset Δ_0 of Δ such that $\cap\{i_\mu(V_\alpha) \mid \alpha \in \Delta_0\} = \emptyset$.

Proof. (a) \Rightarrow (b). Let \mathcal{F} be a maximal filterbase on X . Suppose that \mathcal{F} does not θ_μ -converge to any point of X . Since \mathcal{F} is maximal, \mathcal{F} does not θ_μ -adheres at any point of X . For each $x \in X$, there exists $F_x \in \mathcal{F}$ and μ -open set U_x containing x such

that $F_x \cap c_\mu(U_x) = \emptyset$. Now $\{U_x \mid x \in X\}$ is a μ -open cover for X . By (a), there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of X such that $X = \cup\{c_\mu(U_{x_i}) \mid 1 \leq i \leq n\}$. Since \mathcal{F} is a filterbase, there exists $F \in \mathcal{F}$ such that $F \subset \cap F_{x_i}$. Now, $\cap F_{x_i} = (\cap F_{x_i}) \cap (\cup c_\mu(U_{x_i})) = \cup((\cap F_{x_i}) \cap c_\mu(U_{x_i})) = \emptyset$. Hence $F = \emptyset$, a contradiction.

(b) \Rightarrow (c). Let \mathcal{F} be a filterbase on X . Then there exists a maximal filterbase \mathcal{F}_0 such that $\mathcal{F} \subset \mathcal{F}_0$. By (b), \mathcal{F}_0 θ_μ -converges to some point $x \in X$. For every $F \in \mathcal{F}$ and every $U \in \mu$ containing x , there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset c_\mu(U)$. Now $F_0 \subset c_\mu(U)$ implies that $F_0 \cap F \subset c_\mu(U) \cap F$. Since \mathcal{F}_0 is a filterbase, $F \cap F_0 \neq \emptyset$ and so $F \cap c_\mu(U) \neq \emptyset$. Hence \mathcal{F} θ_μ -adheres at x .

(c) \Rightarrow (d). Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a family of μ -closed subsets of X such that $\cap\{V_\alpha \mid \alpha \in \Delta\} = \emptyset$. Let \mathcal{J} be the family of all finite subsets of Δ . Assume that $A_I = \cap\{i_\mu(V_\alpha) \mid \alpha \in I\} \neq \emptyset$ for every $I \in \mathcal{J}$. Then the family $\mathcal{F} = \{A_I \mid I \in \mathcal{J}\}$ is a filterbase on X . By hypothesis, \mathcal{F} θ_μ -adheres at some point $x \in X$. Since $\{X - V_\alpha \mid \alpha \in \Delta\}$ is a μ -open cover of X , $x \in X - V_\beta$ for some $\beta \in \Delta$. Since $c_\mu(X - V_\beta) \cap i_\mu(V_\beta) = \emptyset$, we have a contradiction to the fact that \mathcal{F} θ_μ -adheres at $x \in X$. Thus $\cap\{i_\mu(V_\alpha) \mid \alpha \in I\} = \emptyset$ for some $I \in \mathcal{J}$.

(d) \Rightarrow (a). Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a cover of X by μ -open sets of X . Then $\{X - V_\alpha \mid \alpha \in \Delta\}$ is a family of μ -closed subsets of X such that $\cap\{X - V_\alpha \mid \alpha \in \Delta\} = \emptyset$. By (d), there exists a finite subset Δ_0 of Δ such that $\cap\{i_\mu(X - V_\alpha) \mid \alpha \in \Delta_0\} = \emptyset$. Hence $X - \cap\{i_\mu(X - V_\alpha) \mid \alpha \in \Delta_0\} = X$ which implies that $\cup\{c_\mu(V_\alpha) \mid \alpha \in \Delta_0\} = X$. Hence X is μ -closed. \square

The following Theorem 2.5 gives a characterization of μ -closed spaces in terms of θ_μ -convergence of a grill. Theorem 2.6 gives a characterization of μ -closed relative to X subsets in terms of θ_μ -convergence of a grill. Theorem 3.4 of [10] gives similar characterization of π -closed relative to X subsets. The following Corollary 2.1 shows that Theorem 2.5 is true for $\mu = \tau$.

Theorem 2.5. *A space (X, τ) is μ -closed if and only if every grill \mathcal{G} on X θ_μ -converges in X .*

Proof. Let \mathcal{G} be a grill in a μ -closed space X . Then by Lemma 1.1, $\text{sec } \mathcal{G}$ is a filter on X . Let $B \in \text{sec } \mathcal{G}$. Then $B \cap G \neq \emptyset$ for every $G \in \mathcal{G}$ which implies that $X - B \notin \mathcal{G}$. Hence $B \in \mathcal{G}$, since \mathcal{G} is a grill. Thus, $\text{sec } \mathcal{G} \subset \mathcal{G}$. By Lemma 1.1, there is an ultrafilter \mathcal{U} on X such that $\text{sec } \mathcal{G} \subset \mathcal{U} \subset \mathcal{G}$. Since X is μ -closed, by Theorem 2.4, \mathcal{U} θ_μ -converges to some point $x \in X$. Then for each μ -open set U containing x , there exists some $G \in \mathcal{U}$ such that $G \subset c_\mu(U)$. Since $c_\mu(U) \in \mathcal{U}$, $c_\mu(U) \in \mathcal{G}$ for every $U \in \mu$ containing x . Hence \mathcal{G} θ_μ -converges to x . Conversely, let \mathcal{U} be an ultrafilter on X . Since every ultrafilter is a grill, \mathcal{U} θ_μ -converges to some point of X . \square

Corollary 2.1. *A space (X, τ) is almost compact if and only if every grill \mathcal{G} on X θ -converges in X .*

Corollary 2.2. *A space (X, τ) is s -closed if and only if every grill \mathcal{G} on X θ_σ -converges in X .*

Corollary 2.3. [10, Theorem 3.3] *A space (X, τ) is p -closed if and only if every grill \mathcal{G} on X θ_π -converges in X .*

Corollary 2.4. *A space (X, τ) is b -closed if and only if every grill \mathcal{G} on X θ_b -converges in X .*

Corollary 2.5. *A space (X, τ) is β -closed if and only if every grill \mathcal{G} on X θ_β -converges in X .*

Theorem 2.6. *A subset A of a space (X, τ) is μ -closed relative to X if and only if every grill \mathcal{G} on X with $A \in \mathcal{G}$, θ_μ -converges to a point in A .*

Proof. Let A be μ -closed relative to X and \mathcal{G} be a grill on X such that $A \in \mathcal{G}$ and \mathcal{G} does not θ_μ -converge to any point of A . Then, for each $x \in A$, there exists $U_x \in \mu$ containing x such that $c_\mu(U_x) \notin \mathcal{G}$. Now, $\{U_x \mid x \in A\}$ is a cover of A by μ -open sets of X . Since A is μ -closed relative to X , there exists a finite subset B of A such that $A \subset \cup\{c_\mu(U_x) \mid x \in B\}$. Since \mathcal{G} is a grill, $\cup\{c_\mu(U_x) \mid x \in B\} \notin \mathcal{G}$. Hence $A \notin \mathcal{G}$, a contradiction. Hence \mathcal{G} θ_μ -converges to some point of A . Conversely, let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a cover of A by μ -open sets of X . If A is not μ -closed relative to X , then for every finite subset Δ_0 of Δ , $A - \cup\{c_\mu(U_\alpha) \mid \alpha \in \Delta_0\} \neq \emptyset$. Let $\mathcal{F} = \{A - \cup c_\mu(U_\alpha) \mid \alpha \in \Delta_0, \Delta_0 \subset \Delta \text{ is finite}\}$. Then \mathcal{F} is a filterbase on X . The family \mathcal{F} can be extended to an ultrafilter \mathcal{M} on X . Then \mathcal{M} is a grill on X . Since for every $F \in \mathcal{F}$, $F \subset A$, we have $A \in \mathcal{M}$. Since \mathcal{U} is a cover of A , for each $x \in A$, there exists $\beta \in \Delta$ such that $x \in U_\beta$. For any $G \in \mathcal{M}$, $G \cap (A - c_\mu(U_\beta)) \neq \emptyset$. Hence $G \not\subset c_\mu(U_\beta)$ for all $G \in \mathcal{M}$ and so \mathcal{M} does not θ_μ -converge to any point of A , a contradiction. □

Theorem 2.7. *Let (X, τ) be a space such that every grill \mathcal{G} on X with the property $\cap\{c_{\theta_\mu}(G_i) \mid 1 \leq i \leq n\} \neq \emptyset$ for every finite subfamily $\{G_1, G_2, \dots, G_n\}$ of \mathcal{G} , θ_μ -adheres in X , then X is a μ -closed space.*

Proof. Let \mathcal{U} be any ultrafilter on X . Then \mathcal{U} is a grill on X . Also, for each finite subcollection $\{U_1, U_2, \dots, U_n\}$ of \mathcal{U} , $\cap\{U_i \mid 1 \leq i \leq n\} \neq \emptyset$. Since

$$\cap\{U_i \mid 1 \leq i \leq n\} \subset \cap\{c_{\theta_\mu}(U_i) \mid 1 \leq i \leq n\}, \quad \cap\{c_{\theta_\mu}(U_i) \mid 1 \leq i \leq n\} \neq \emptyset.$$

Hence by hypothesis, \mathcal{U} θ_μ -adheres in X . By Theorem 2.4, X is μ -closed. □

A grill \mathcal{G} on X is said to be θ_μ -linked if for any two members $A, B \in \mathcal{G}$, $c_{\theta_\mu}(A) \cap c_{\theta_\mu}(B) \neq \emptyset$. \mathcal{G} is said to be θ_μ -conjoint if for every finite subfamily $\{A_1, A_2, \dots, A_n\}$ of \mathcal{G} , $i_\mu(\cap\{c_{\theta_\mu}(A_i) \mid 1 \leq i \leq n\}) \neq \emptyset$. Clearly, every θ_μ -conjoint grill is θ_μ -linked. If $\mu = \pi$, we have the $p(\theta)$ -linked grill [10] and $p(\theta)$ -conjoint grill

[10]. Example 2.12 of [10] gave an example of a $p(\theta)$ -linked grill which is not a $p(\theta)$ -conjoint grill.

Theorem 2.8. *Let (X, τ) be a μ -closed space. Then every θ_μ -conjoint grill θ_μ -adheres in X .*

Proof. Let \mathcal{G} be a θ_μ -conjoint grill on a μ -closed space X . Since $c_{\theta_\mu}(A)$ is μ -closed for every $A \subset X$, $\{c_{\theta_\mu}(A) \mid A \in \mathcal{G}\}$ is a collection of μ -closed sets in X . Since \mathcal{G} is θ_μ -conjoint, for any finite subfamily $\{A_1, A_2, \dots, A_n\}$ of \mathcal{G} , $i_\mu(\cap\{c_{\theta_\mu}(A_i) \mid 1 \leq i \leq n\}) \neq \emptyset$ and so $\cap\{i_\mu(c_{\theta_\mu}(A_i)) \mid 1 \leq i \leq n\} \neq \emptyset$. Hence by Theorem 2.4, $\cap\{c_{\theta_\mu}(A) \mid A \in \mathcal{G}\} \neq \emptyset$ and so there exists $x \in X$ such that $x \in c_{\theta_\mu}(A)$ for every $A \in \mathcal{G}$ which implies that $A \in \mathcal{G}(\theta_\mu, x)$ for all $A \in \mathcal{G}$ which in turn implies that $\mathcal{G} \subset \mathcal{G}(\theta_\mu, x)$. Therefore, \mathcal{G} θ_μ -adheres at x , by Theorem 2.2. \square

The following Lemma 2.1 will be useful to prove Theorem 2.9 below.

Lemma 2.1. *Let (X, τ) be a space and $\mu \in \{\sigma, b, \beta\}$ and $A \subset X$. Then $c_{\theta_\mu}(c_{\theta_\mu}(A)) = c_{\theta_\mu}(A)$.*

Proof. For $\mu = \beta$, the proof is in [12]. For $\mu = b$, the proof is in [13]. For $\mu = \sigma$, the proof is as follows. Clearly, $c_{\theta_\mu}(A) \subset c_{\theta_\mu}(c_{\theta_\mu}(A))$. If $x \notin c_{\theta_\mu}(A)$, then there exists a semiregular set V containing x such that $V \cap A = \emptyset$. Since V is semiregular, $V \cap c_{\theta_\mu}(A) = \emptyset$ and so $x \notin c_{\theta_\mu}(c_{\theta_\mu}(A))$. \square

Theorem 2.9. *Let (X, τ) be a μ -closed space where $\mu \in \{\sigma, b, \beta\}$. Then every grill \mathcal{G} on X , with the property that $\cap\{c_{\theta_\mu}(G_i) \mid 1 \leq i \leq n\} \neq \emptyset$ for every finite subfamily $\{G_1, G_2, \dots, G_n\}$ of \mathcal{G} , θ_μ -adheres in X .*

Proof. Let X be a μ -closed space. $\mathcal{G} = \{G_\alpha \mid \alpha \in \Delta\}$ be a grill on X with the property $\cap\{c_{\theta_\mu}(G_\alpha) \mid \alpha \in \Delta_0\} \neq \emptyset$ for every finite subset Δ_0 of Δ . Consider the family $\mathcal{F} = \{\cap\{c_{\theta_\mu}(G_\alpha) \mid \alpha \in \Delta_0\} \mid \Delta_0 \subset \Delta \text{ is finite}\}$. Then \mathcal{F} is a filterbase on X . Since X is μ -closed, \mathcal{F} θ_μ -adheres at some $x \in X$ which implies that $x \in c_{\theta_\mu}(c_{\theta_\mu}(G))$ for every $G \in \mathcal{G}$ which in turn implies that $x \in c_{\theta_\mu}(G)$ for every $G \in \mathcal{G}$. Hence $\mathcal{G} \subset \mathcal{G}(\theta_\mu, x)$ and so the proof follows from Theorem 2.2. \square

Corollary 2.6. *Let (X, τ) be a space where $\mu \in \{\sigma, b, \beta\}$. Then the following are equivalent.*

- (a) *Every grill \mathcal{G} on X , with the property that $\cap\{c_{\theta_\mu}(G_i) \mid 1 \leq i \leq n\} \neq \emptyset$ for every finite subfamily $\{G_1, G_2, \dots, G_n\}$ of \mathcal{G} , θ_μ -adheres in X .*
- (b) *X is μ -closed.*

Proof. (a) \Rightarrow (b). The proof follows from Theorem 2.7.

(b) \Rightarrow (a). The proof follows from Theorem 2.9. \square

A space X is μ compact where $\mu \in \{\tau, \alpha, \sigma, \pi, b, \beta\}$ if every cover \mathcal{U} of X by μ -open sets of X has a finite subcover. Every μ -compact space is μ -closed. A space X is μ regular if for each $x \in X$ and each $U \in \mu$ containing x , there exists a μ -open set V containing x such that $c_\mu(V) \subset U$. A space X is θ_μ -regular if every grill on X which θ_μ -converges must μ -converge. A grill \mathcal{G} on X is said to be μ -converge to a point $x \in X$ if $\mu(x) \subset \mathcal{G}$, where $\mu(x)$ denote the family of all μ -open sets containing x .

Theorem 2.10. *A space (X, τ) is μ -compact if and only if every grill μ -converges.*

Proof. Let \mathcal{G} be a grill on a μ -compact space X such that \mathcal{G} does not μ -converge to any point $x \in X$. Then for each $x \in X$, there exists a μ -open set U_x containing x with $U_x \notin \mathcal{G}$. Since $\{U_x \mid x \in X\}$ is a cover of the μ -compact X by μ -open sets, there exist finitely many points $\{x_1, x_2, \dots, x_n\}$ in X such that $X = \cup\{U_{x_i} \mid 1 \leq i \leq n\}$. Since $X \in \mathcal{G}$, there exists U_{x_i} such that $U_{x_i} \in \mathcal{G}$ for $1 \leq i \leq n$, a contradiction. Conversely, let every grill on X μ -converge. Suppose X is not μ -compact. Then there exists a cover \mathcal{U} of X by μ -open sets of X having no finite subcover. Then $\mathcal{F} = \{X - \cup \mathcal{U}_0 \mid \mathcal{U}_0 \text{ is a finite subcollection of } \mathcal{U}\}$ is a filterbase on X . Then \mathcal{F} is contained in an ultrafilter \mathcal{G} and so \mathcal{G} is a grill on X . By hypothesis, \mathcal{G} μ -converges to some $x \in X$. Then for some $U \in \mathcal{U}$, $x \in U$ and hence $U \in \mathcal{G}$. But $X - U \in \mathcal{F} \subset \mathcal{G}$. Hence U and $X - U$ both belong to \mathcal{G} , which is an ultrafilter, a contradiction. \square

Theorem 2.11. *A μ -compact space X is μ -closed. The converse holds if X is θ_μ -regular.*

Proof. Suppose \mathcal{G} is a grill on a μ -closed space X . Then \mathcal{G} θ_μ -converges in X , by Theorem 2.5. Since X is θ_μ -regular, \mathcal{G} μ -converges. By Theorem 2.10, X is μ -compact. \square

Theorem 2.12. *Every μ -regular space is θ_μ -regular.*

Proof. Let \mathcal{G} be a grill on a μ -regular space X . Suppose \mathcal{G} θ_μ -converges to a point $x \in X$. For each $U \in \mu$ containing x , there exists a $V \in \mu$ containing x such that $c_\mu(V) \subset U$. By hypothesis, $c_\mu(V) \in \mathcal{G}$ and so $U \in \mathcal{G}$. Hence \mathcal{G} μ -converges to x . Thus, X is θ_μ -regular. \square

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