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NULL LAGRANGIAN FORMS ON 2ND ORDER JET BUNDLES

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Invited paper to celebrate Professor Constantin Udrişte,
on the occasion of his seventies

Abstract. The paper analyzes the null Lagrangian forms on the second order jet bundles, extending the results in [11]. The term *null Lagrangian* is associated to a Lagrangian whose Euler-Lagrange PDEs are identically satisfied. §1 refers to the Lagrange 1-forms of the second order in both cases, single and multi-time. §2 finds the Euler-Lagrange PDEs associated to path independent curvilinear functionals. §3 presents equivalent conditions for null Lagrangians in the single-time and multi-time cases. §4 deals with null Lagrange 1-forms and total derivatives.

1. INTRODUCTION

Null Lagrangian ([1], [3], [4], [10]) refers to a Lagrangian L whose integral over any domain $\Omega \subset \mathbb{R}^m$ can be reduced, using integration by parts, to an integral over the boundary $\partial\Omega$. As an example, we may give the Jacobian determinant $L = J(x)(t)$ associated to a diffeomorphism $x\colon \mathbb{R}^m \to \mathbb{R}^m$, which is used in the volume functional

$$V(x(\cdot)) = \int_\Omega J(x)(t)dt^1 \cdots dt^m.$$

The importance of null Lagrangians is highlighted by of recent research in the calculus of variations (polyconvex energy integrals), nonlinear PDEs (compensated compactness), geometric function theory (quasiconformal deformations) and some areas of applied mathematics: nonlinear elasticity, material science and crystals (rank-one connections). Adding null Lagrangian to any integrand does not affect the variational Euler-Lagrange equation, [2].

2. LAGRANGE 1-FORMS OF THE 2ND ORDER AND THEIR PRIMITIVES

2.1. **Lagrange 1-forms of the 2nd order and their primitives as simple integrals.** A Lagrange 1-form of the second order on the jets space $J^2(\mathbb{R}, M)$ with

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the form

$$\begin{aligned}\omega &= L(t,x(t),\dot{x}(t),\ddot{x}(t))dt + M_i(t,x(t),\dot{x}(t),\ddot{x}(t))dx^i \\ &+N_i(t,x(t),\dot{x}(t),\ddot{x}(t))d\dot{x}^i + P_i(t,x(t),\dot{x}(t),\ddot{x}(t))d\ddot{x}^i,\end{aligned}$$

where L, M_i, N_i and P_i are Lagrangians of the second order, has the pullback

$$x^*\omega = (L + M_i\dot{x}^i + N_i\ddot{x}^i + P_i\dddot{x}^i)dt,$$

a Lagrange 1-form of the third order on M. The coefficient

$$L + M_i\dot{x}^i + N_i\ddot{x}^i + P_i\dddot{x}^i$$

is a second order Lagrangian, which is linear in third order derivatives. To the form ω we attach the Pfaff equation $\omega = 0$ and the differential equation

$$L + M_i\dot{x}^i + N_i\ddot{x}^i + P_i\dddot{x}^i = 0.$$

There are two different ways to write the primitive of a Lagrange 1-form of the form $L(t,x(t),\dot{x}(t),\ddot{x}(t))dt$: the first one, as a definite integral,

$$\phi(t) = \int_{t_0}^{t} L(s,x(s),\dot{x}(s),\ddot{x}(t))ds, \quad \phi(t_0) = 0$$

and the second one, as a differential equation

$$\dot{\phi}(t) = L(t,x(t),\dot{x}(t),\ddot{x}(t)), \quad \phi(t_0) = 0.$$

If there is a Lagrangian-like primitive

$$K(t,x(t),\dot{x}(t),\ddot{x}(t)) = \int_{t_0}^{t} L(s,x(s),\dot{x}(s),\ddot{x}(s))ds, \quad K(t_0,x(t_0),\dot{x}(t_0),\ddot{x}(t)) = 0,$$

or $\dfrac{d}{dt}K = L$, then it follows the relation

$$\begin{aligned}&\frac{\partial K}{\partial t}(t,x(t),\dot{x}(t),\ddot{x}(t)) + \frac{\partial K}{\partial x^i}(t,x(t),\dot{x}(t),\ddot{x}(t))\dot{x}^i(t) + \frac{\partial K}{\partial \dot{x}^i}(t,x(t),\dot{x}(t),\ddot{x}(t))\ddot{x}^i(t) \\ &\qquad + \frac{\partial K}{\partial \ddot{x}^i}(t,x(t),\dot{x}(t),\ddot{x}(t))\dddot{x}^i(t) = L(t,x(t),\dot{x}(t),\ddot{x}(t)).\end{aligned}$$

This relation is fulfilled by any function $x(t)$ if and only if $\dfrac{\partial K}{\partial \ddot{x}^i} = 0$, therefore $K = K(t,x(t),\dot{x}(t))$, and L is a linear function in $\ddot{x}^i$.

2.2. **Lagrange 1-forms of the 2nd order and their primitives as curvilinear integrals.** A Lagrange 1-form of the second order on the jets space $J^2(T,M)$ of the form

$$\begin{aligned}\omega &= L_\alpha(t,x(t),x_\gamma(t),x_{\mu\nu}(t))dt^\alpha + M_i(t,x(t),x_\gamma(t),x_{\mu\nu}(t))dx^i \\ &+N_i^\beta(t,x(t),x_\gamma(t),x_{\mu\nu}(t))dx^i_\beta + P_i^{\alpha\beta}(t,x(t),x_\gamma(t),x_{\mu\nu}(t))dx^i_{\alpha\beta},\end{aligned}$$

where L_α, M_i, N_i^β and $P_i^{\alpha\beta}$ are Lagrangians of the second order, $x_\gamma(t) = \dfrac{\partial x}{\partial t^\gamma}(t)$ and $x_{\mu\nu}(t) = \dfrac{\partial^2 x}{\partial t^\mu \partial t^\nu}(t)$, has the pullback

$$x^*\omega = \left(L_\alpha + M_i x_\alpha^i + N_i^\beta x_{\beta\alpha}^i + P_i^{\beta\gamma} x_{\alpha\beta\gamma}^i\right) dt^\alpha$$

which is a Lagrange 1-form of the third order on M. The coefficients

$$L_\alpha + M_i x_\alpha^i + N_i^\beta x_{\beta\alpha}^i + P_i^{\beta\gamma} x_{\alpha\beta\gamma}^i$$

are third order Lagrangians, which are linear in the third order derivatives. To the form ω one attaches the Pfaff equation $\omega = 0$ and the partial differential equations

$$L_\alpha + M_i x_\alpha^i + N_i^\beta x_{\beta\alpha}^i + P_i^{\beta\gamma} x_{\alpha\beta\gamma}^i = 0.$$

Let $L_\beta(t, x(t), x_\gamma(t), x_{\mu\nu}(t))dt^\beta$ be a closed Lagrange 1-form (completely integrable), that is $D_\beta L_\alpha = D_\alpha L_\beta$.

A closed 1-form in a simple-connected domain is an exact one. Its primitive can be expressed as a curvilinear integral

$$\phi(t) = \int_{\Gamma_{t_0,t}} L_\alpha\left(s, x(s), x_\gamma(s), x_{\mu\nu}(s)\right) ds^\alpha, \quad \phi(t_0) = 0,$$

or as a system of PDEs,

$$\frac{\partial \phi}{\partial t^\alpha}(t) = L_\alpha\left(t, x(t), x_\gamma(t), x_{\mu\nu}(t)\right), \quad \phi(t_0) = 0.$$

If would exist a Lagrangian-like primitive

$$L(t, x(t), x_\gamma(t), x_{\mu\nu}(t)) = \int_{\Gamma_{t_0,t}} L_\alpha\left(s, x(s), x_\gamma(s), x_{\mu\nu}(s)\right) ds^\alpha,$$

$$L\left(t_0, x(t_0), x_\gamma(t_0), x_{\mu\nu}(t_0)\right) = 0$$

or $D_\alpha L = L_\alpha$ (the foregoing pullback is the given closed 1-form),

$$\frac{\partial L}{\partial t^\beta} + \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L}{\partial x_\gamma^i}\frac{\partial x_\gamma^i}{\partial t^\beta} + \frac{\partial L}{\partial x_{\mu\nu}^i}\frac{\partial x_{\mu\nu}^i}{\partial t^\beta} = L_\beta,$$

relation which can be understood as a completely integrable system of PDEs (of the second order) with the unknown function $x(t)$, too.

Any smooth *Lagrangian* $L\left(t, x(t), x_\gamma(t), x_{\mu\nu}(t)\right)$, $t \in \mathbb{R}_+^m$, produces two smooth closed (completely integrable) 1-forms:

- the differential

$$dL = \frac{\partial L}{\partial t^\gamma} dt^\gamma + \frac{\partial L}{\partial x^i} dx^i + \frac{\partial L}{\partial x_\gamma^i} dx_\gamma^i + \frac{\partial L}{\partial x_{\mu\nu}^i} dx_{\mu\nu}^i$$

of components $\left(\dfrac{\partial L}{\partial t^\gamma}, \dfrac{\partial L}{\partial x^i}, \dfrac{\partial L}{\partial x_\gamma^i}, \dfrac{\partial L}{\partial x_{\mu\nu}^i}\right)$, with respect to the corresponding basis $(dt^\gamma,\, dx^i,\, dx_\gamma^i,\, dx_{\mu\nu}^i)$;

- the restriction of dL to $(t, x(t), x_\gamma(t), x_{\mu\nu}(t))$, that is the pullback

$$dL\Big|_{(t,x(t),x_\gamma(t),x_{\mu\nu}(t))} = \left(\frac{\partial L}{\partial t^\beta} + \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta} + \frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_{\mu\nu}}{\partial t^\beta}\right) dt^\beta,$$

of components

$$D_\beta L = \frac{\partial L}{\partial t^\beta}(t, x(t), x_\gamma(t), x_{\mu\nu}(t)) + \frac{\partial L}{\partial x^i}(t, x(t), x_\gamma(t), x_{\mu\nu}(t))\frac{\partial x^i}{\partial t^\beta}(t)$$
$$+ \frac{\partial L}{\partial x^i_\gamma}(t, x(t), x_\gamma(t), x_{\mu\nu}(t))\frac{\partial x^i_\gamma}{\partial t^\beta}(t) + \frac{\partial L}{\partial x^i_{\mu\nu}}(t, x(t), x_\gamma(t), x_{\mu\nu}(t))\frac{\partial x^i_{\mu\nu}}{\partial t^\beta}(t),$$

with respect to the basis dt^β (for other significant ideas, see [5]÷[12]).

3. THE EXTREMALS OF THE FUNCTIONALS REPRESENTED BY PATH-INDEPENDENT CURVILINEAR INTEGRALS

Let Γ_{t_0,t_1} be an arbitrary piecewise C^1-curve joining the diagonal opposite points t_0 and t_1 of the parallelepiped $\Omega_{t_0,t_1} \subset \mathbb{R}^m_+$. Our aim is to find an m-sheet $x^*(\cdot) : \Omega_{t_0,t_1} \to \mathbb{R}^n$ which minimize *the functional represented by path-independent curvilinear integral (action)*

$$J(x(\cdot)) = \int_{\Gamma_{0,t_0}} L_\beta(t, x(t), x_\gamma(t), x_{\mu\nu}(t))dt^\beta,$$

where the function $x(\cdot)$ satisfies the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$, $x_\gamma(t_0) = x_{\gamma_0}, x_\gamma(t_1) = x_{\gamma_1}$ or $x(t)\big|_{\partial\Omega_{t_0,t_1}} = \text{given}, x_\gamma(t)\big|_{\partial\Omega_{t_0 t_1}} = \text{given}$, and variations (functions) constrained by boundary conditions and by closeness conditions (completely integrability) for the Lagrange 1-form.

Fundamental problem. Characterize the function $x^*(\cdot)$ which solves the variational problem associated to functional J.

Theorem 3.1. *Suppose that there is a Lagrangian $L(t, x(t), x_\gamma(t), x_{\mu\nu}(t))$ with the property $D_\beta L = L_\beta$.*

1) *If the m-sheet $x^*(\cdot)$ is an extremal for L, then it is an extremal for the differential dL also.*

2) *If the m-sheet $x^*(\cdot)$ minimizes the functional $J(x(\cdot))$, then $x^*(\cdot)$ is a solution of the multi-time PDEs*

$$\frac{\partial L}{\partial x^i} - D_\gamma\left(\frac{\partial L}{\partial x^i_\gamma}\right) + D_\mu D_\nu\left(\frac{\partial L_\beta}{\partial x^i_{\mu\nu}}\right) = a_i,\ i = \overline{1, n} \tag{3.1}$$

which satisfies boundary conditions, where a_i are arbitrary constants.

The second part of this theorem shows that if the system (3.1) of PDEs has solutions, then the minimizing function for the functional J (supposing it exists) lies between these solutions.

Proof. If we have in mind the following equalities

$$0=d\left(\frac{\partial L}{\partial x^i}-D_\gamma\left(\frac{\partial L}{\partial x^i_\gamma}\right)+D_\gamma D_\beta\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\right)\right)=\frac{\partial (dL)}{\partial x^i}-D_\gamma\left(\frac{\partial (dL)}{\partial x^i_\gamma}\right)+D_\gamma D_\beta\left(\frac{\partial (dL)}{\partial x^i_{\mu\nu}}\right),$$

both relations are obvious. □

Theorem 3.2. *If the m-sheet $x^*(\cdot)$ minimizes the functional $J(x(\cdot))$, then $x^*(\cdot)$ is a solution of the multi-time PDE*

$$\frac{\partial L_\beta}{\partial x^i}-D_\gamma\left(\frac{\partial L_\beta}{\partial x^i_\gamma}\right)+D_\mu D_\nu\left(\frac{\partial L_\beta}{\partial x^i_{\mu\nu}}\right)=0,\ \beta=\overline{1,m},\ i=\overline{1,n} \tag{3.2}$$

which satisfies boundary conditions.

The theorem shows that if the system (3.2) of PDEs has solutions, then the minimizing function for the functional J (supposing it exists) lies between these solutions.

Proof. Let us consider that $x(\cdot)$ is a solution of the foregoing problem, satisfying the complete integrability conditions of the 1-form $L=L_\beta(t,x(t),x_\gamma(t),x_{\mu\nu}(t))dt^\beta$, that is $\dfrac{\partial L_\beta}{\partial L_\alpha}=\dfrac{\partial L_\alpha}{\partial L_\beta}$, for all $\alpha,\beta=\overline{1,m}$, $\alpha\neq\beta$. Construct, near $x(t)$, another function having the form $x(t)+\varepsilon h(t)$, $h(t_0)=0$, $h(t_1)=0$, $h_\gamma(t_0)=0$, $h_\gamma(t_1)=0$, where ε is a "small" parameter, and h is a "small" variation. The functional becomes a function of ε,

$$J(\varepsilon)=\int_{\Gamma_{t_0,t_1}}L_\beta\left(t,x(t)+\varepsilon h(t),x_\gamma(t)+\varepsilon h_\gamma(t),x_{\mu\nu}(t)+\varepsilon h_{\mu\nu}(t)\right)dt^\beta.$$

Suppose that the variation h satisfies the complete integrability conditions of the 1-form

$$L_\beta\left(t,x(t)+\varepsilon h(t),x_\gamma(t)+\varepsilon h_\gamma(t),x_{\mu\nu}(t)+\varepsilon h_{\mu\nu}(t)\right)dt^\beta,$$

which shows that the set of the functions $h(t)$ is a vector space and the set of the functions $x(t)+\varepsilon h(t)$ is an affine space.

The following relations hold true:

$$0=\frac{d}{d\varepsilon}J(\varepsilon)\Big|_{\varepsilon=0}=\int_{\Gamma_{t_0,t_1}}\left(\frac{\partial L_\beta}{\partial x^j}h^j+\frac{\partial L_\beta}{\partial x^j_\gamma}h^j_\gamma+\frac{\partial L_\beta}{\partial x^j_{\mu\nu}}h^j_{\mu\nu}\right)dt^\beta$$

$$=\int_{\Gamma_{t_0,t_1}}\left(\frac{\partial L_\beta}{\partial x^j}h^j+D_\gamma\left(\frac{\partial L_\beta}{\partial x^j_\gamma}h^j\right)-D_\gamma\left(\frac{\partial L_\beta}{\partial x^j_\gamma}\right)h^j\right.$$
$$\left.+D_\nu\left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}}h^j_\mu\right)-D_\nu\left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}}\right)h^j_\mu\right)dt^\beta$$

$$= \int_{\Gamma_{t_0 t_1}} \left(\left(\left(\frac{\partial L}{\partial x^j} - D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_\gamma} \right) + D_\mu D_\nu \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} \right) \right) h^j \right.\right.$$

$$\left.\left. + \; D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_\gamma} h^j \right) - D_\mu \left(D_\nu \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} \right) h^j \right) \right) + D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} h^j_\mu \right) \right) dt^\beta,$$

where D_γ is the total derivative operator. By adding the hypotheses

$$D_\gamma \left(h^i \frac{\partial L_\beta}{\partial x^i_\gamma} \right) = D_\beta \left(h^i \frac{\partial L_\gamma}{\partial x^i_\gamma} \right),$$

$$D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} h^j_\mu \right) = D_\beta \left(\frac{\partial L_\gamma}{\partial x^j_{\mu\nu}} h^j_\mu \right)$$

and

$$D_\mu \left(D_\nu \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} \right) h^j \right) = D_\beta \left(D_\nu \left(\frac{\partial L_\mu}{\partial x^j_{\mu\nu}} \right) h^j \right),$$

the vector space of the variations $h(t)$ is restricted to a subspace.

We obtain

$$\int_{\Gamma_{t_0,t_1}} D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_\gamma} h^j \right) dt^\beta = \int_{\Gamma_{t_0,t_1}} D_\beta \left(\frac{\partial L_\gamma}{\partial x^j_\gamma} h^j \right) dt^\beta = \frac{\partial L_\gamma}{\partial x^j_\gamma} h^j \Big|_{t_0}^{t_1},$$

$$\int_{\Gamma_{t_0,t_1}} D_\gamma \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} h^j_\mu \right) dt^\beta = \int_{\Gamma_{t_0,t_1}} D_\beta \left(\frac{\partial L_\gamma}{\partial x^j_{\mu\nu}} h^j_\mu \right) dt^\beta = \frac{\partial L_\gamma}{\partial x^j_{\mu\nu}} h^j_\mu \Big|_{t_0}^{t^1}$$

and

$$\int_{\Gamma_{t_0,t_1}} D_\mu \left(D_\nu \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} \right) h^j \right) dt^\beta = D_\nu \left(\frac{\partial L_\mu}{\partial x^j_{\mu\nu}} \right) h^j \Big|_{t_0}^{t_1}.$$

These terms vanish since $h(t_0) = 0$, $h(t_1) = 0$, $h_\gamma(t_0) = 0$, $h_\gamma(t_1) = 0$. Therefore,

$$0 = \int_{\Gamma_{t_0,t_1}} \left(\frac{\partial L_\beta}{\partial x^j} - D_\gamma \frac{\partial L_\beta}{\partial x^j_\gamma} + D_\mu D_\nu \left(\frac{\partial L_\beta}{\partial x^j_{\mu\nu}} \right) \right) h^j \, dt^\beta.$$

The curve Γ_{t_0,t_1} is an arbitrary one, hence we have obtained the conclusion of the theorem. □

For other advances regarding path independent curvilinear functionals, we address the reader to [5]÷[12].

4. EQUIVALENT CONDITIONS FOR NULL LAGRANGIANS

4.1. **The single-time case: null Lagrangians and the total derivative.** There are Lagrangians whose Euler-Lagrange ODE are identically satisfied. In these situations, each curve $x(t)$ may be an extremal. Such kind of Lagrangian is called *null Lagrangian.* The definition of null Lagrangians does not depend on the coordinate system. Generally speaking, a Lagrangian having the form

$$L(t, x(t), \dot{x}(t), \ddot{x}(t)) = D_t K(t, x(t), \dot{x}(t))$$

is a null one because

$$\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t))dt = K(t, x(t), \dot{x}(t))\Big|_{t_0}^{t_1},$$

that is the integral depends only on the values of the function $x(t)$ taken on the boundary and it is not affected by variations $h(t)$. Fortunately, all null Lagrangians have this form.

Theorem 4.1. *A Lagrangian $L(t, x(t), \dot{x}(t), \ddot{x}(t))$, $t \in \mathbb{R}_+$, is a null one if and only if it is a total derivative.*

Proof. If L is a total derivative, it is clear that $L = D_t K$ is a null Lagrangian.

To prove the converse, suppose that the Euler-Lagrange equations associated to the Lagrangian $L(t, x(t), \dot{x}(t), \ddot{x}(t))$, $t \in \mathbb{R}_+$, are identically satisfied. Construct the function $f(\varepsilon) = L(t, \varepsilon x(t), \varepsilon \dot{x}(t), \varepsilon \ddot{x}(t))$ and denote $u(t) = \varepsilon x(t)$. The derivative of Lagrangian L

$$\frac{d}{d\varepsilon} f = x^i \frac{\partial L}{\partial u^i} + \dot{x}^i \frac{\partial L}{\partial \dot{u}^i} + \ddot{x}^i \frac{\partial L}{\partial \ddot{u}^i},$$

becomes, using the total derivative of a product,

$$\begin{aligned} \frac{d}{d\varepsilon} f &= x^i \left(\frac{\partial L}{\partial u^i} - D_t \left(\frac{\partial L}{\partial \dot{u}^i} \right) + D_t D_t \frac{\partial L}{\partial \ddot{u}^i} \right) + D_t \left(x^i \frac{\partial L}{\partial \dot{u}^i} \right) - D_t \left(x^i D_t \left(\frac{\partial L}{\partial \ddot{u}^i} \right) \right) \\ &+ D_t \left(\dot{x}^i \frac{\partial L}{\partial \ddot{u}^i} \right) = D_t \left(x^i \frac{\partial L}{\partial \dot{u}^i} - x^i D_t \left(\frac{\partial L}{\partial \ddot{u}^i} \right) + \dot{x}^i \frac{\partial L}{\partial \ddot{u}^i} \right). \end{aligned}$$

By integration, we have

$$L(t, x(t), \dot{x}(t), \ddot{x}(t)) - L(t, 0, 0, 0) = D_t \int_0^1 \left(x^i \frac{\partial L}{\partial \dot{u}^i} - x^i D_t \left(\frac{\partial L}{\partial \ddot{u}^i} \right) + \dot{x}^i \frac{\partial L}{\partial \ddot{u}^i} \right) d\varepsilon.$$

Moreover, there is a function k with $D_t k = L(t, 0, 0, 0)$. Therefore,

$$L(t, x(t), \dot{x}(t), \ddot{x}(t)) = D_t k + D_t K,$$

where $K = \int_0^1 \left(x^i \frac{\partial L}{\partial \dot{u}^i} - x^i D_t \left(\frac{\partial L}{\partial \ddot{u}^i} \right) + \dot{x}^i \frac{\partial L}{\partial \ddot{u}^i} \right) d\varepsilon.$ □

4.2. **The multi-time case: null Lagrangians and the total divergence.** If a Lagrangian has the property that the corresponding Euler-Lagrange equations are identically satisfied, each m-sheet $x(t)$ may be an extremal one. Such a Lagrangian is called *null Lagrangian.* The definition of null Lagrangians does not depend on the coordinate system.

Let be $S = \partial\Omega$ and n the normal unit vector to S. If a Lagrangian L can be written as a total divergence, $L(t, x(t), x_\gamma(t), x_{\mu\nu}(t)) = \mathrm{Div}\, P(t, x(t), x_\gamma(t))$, then by the theorem of the divergence, we obtain

$$\int_\Omega L\, dt^1 \dots dt^m = \int_\Omega \mathrm{Div}\, P\, dt^1 \dots dt^m = \int_{\partial\Omega} P \cdot n\, dS.$$

The Euler-Lagrange equations are identically satisfied, due to the fact that the functional depends on the values of the function $x(t)$ taken on the boundary only and it is not affected by variations $h(t)$. Hence, the Lagrangian $L = \mathrm{Div}\, P$ is a null one. Fortunately, only the Lagrangians with a total divergence are null Lagrangians.

Theorem 4.2. *A Lagrangian* $L(t, x(t), x_\gamma(t), x_{\mu\nu}(t))$, $t \in \mathbb{R}^m_+$, *is a null one if and only if it is a total divergence.*

Proof. Suppose that the Euler-Lagrange equations associated to $L(t, x(t), x_\gamma(t), x_{\mu\nu})$, $t \in \mathbb{R}_+$, are identically satisfied. Take the function $f(\varepsilon) = L(t, \varepsilon x(t), \varepsilon x_\gamma(t), \varepsilon x_{\mu\nu}(t))$ and denoting by $u(t) = \varepsilon x(t)$, we obtain

$$\frac{d}{d\varepsilon} f = x^i \frac{\partial L}{\partial u^i} + x^i_\gamma \frac{\partial L}{\partial u^i_\gamma} + x^i_{\mu\nu} \frac{\partial L}{\partial u^i_{\mu\nu}}.$$

It follows

$$\begin{aligned} \frac{d}{d\varepsilon} f &= x^i \left(\frac{\partial L}{\partial u^i} - D_\gamma \left(\frac{\partial L}{\partial u^i_\gamma} \right) + D_\mu D_\nu \left(\frac{\partial L}{\partial u^i_{\mu\nu}} \right) \right) + D_\gamma \left(x^i \frac{\partial L}{\partial u^i_\gamma} \right) \\ &\quad + D_\nu \left(x^i_\mu \frac{\partial L}{\partial u^i_{\mu\nu}} \right) - D_\mu \left(x^i D_\nu \left(\frac{\partial L}{\partial u^i_{\mu\nu}} \right) \right) \\ &= D_\gamma \left(x^i \frac{\partial L}{\partial u^i_\gamma} + x^i_\mu \frac{\partial L}{\partial u^i_{\mu\gamma}} - x^i D_\nu \left(\frac{\partial L}{\partial u^i_{\gamma\nu}} \right) \right). \end{aligned}$$

By integration, we get

$$\begin{aligned} &L(t, x(t), x_\gamma(t), x_{\mu\nu}(t)) - L(t, 0, 0, 0) \\ &\qquad = D_\gamma \int_0^1 \left(x^i \frac{\partial L}{\partial u^i_\gamma} + x^i_\mu \frac{\partial L}{\partial u^i_{\mu\gamma}} - x^i D_\nu \left(\frac{\partial L}{\partial u^i_{\gamma\nu}} \right) \right) d\varepsilon. \end{aligned}$$

Moreover, there is a vector p, such that $\mathrm{div}\, p = L(t, 0, 0, 0)$. This implies

$$L = \mathrm{div}\, p + \mathrm{div}\, P,$$

where $P = (P^\gamma)$, $P^\gamma = \displaystyle\int_0^1 x^i \left(\frac{\partial L}{\partial u^i_\gamma} + x^i_\mu \frac{\partial L}{\partial u^i_{\mu\gamma}} - x^i D_\nu \left(\frac{\partial L}{\partial u^i_{\gamma\nu}} \right) \right) d\varepsilon.$ □

Proposition 4.1. *A null Lagrangian L is the curvilinear primitive of the closed Lagrange 1-form*

$$\left(\frac{\partial L}{\partial t^\alpha}+D_\gamma\left(\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i}{\partial t^\alpha}\right)+D_\mu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_\nu}{\partial t^\alpha}\right)-D_\mu\left(D_\nu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\right)\frac{\partial x^i}{\partial t^\alpha}\right)\right)dt^\alpha.$$

Proof. The following equalities hold true:

$$\begin{aligned} dL &= \left(\frac{\partial L}{\partial t^\alpha}+\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\alpha}+\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\alpha}+\frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_{\mu\nu}}{\partial t^\alpha}\right)dt^\alpha \\ &= \frac{\partial L}{\partial t^\alpha}dt^\alpha+\left(\frac{\partial L}{\partial x^i}-D_\gamma\left(\frac{\partial L}{\partial x^i_\gamma}\right)+D_\mu D_\nu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\right)\right)\frac{\partial x^i}{\partial t^\alpha}dt^\alpha \\ &\quad +D_\gamma\left(\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i}{\partial t^\alpha}\right)dt^\alpha+D_\mu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_\nu}{\partial t^\alpha}\right)dt^\alpha-D_\mu\left(D_\nu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\right)\frac{\partial x^i}{\partial t^\alpha}\right)dt^\alpha \\ &= \left(\frac{\partial L}{\partial t^\alpha}+D_\gamma\left(\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i}{\partial t^\alpha}\right)+D_\mu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_\nu}{\partial t^\alpha}\right)-D_\mu\left(D_\nu\left(\frac{\partial L}{\partial x^i_{\mu\nu}}\right)\frac{\partial x^i}{\partial t^\alpha}\right)\right)dt^\alpha. \end{aligned}$$

□

5. NULL LAGRANGE FORMS AND TOTAL DERIVATIVES

There exist multi-time closed Lagrange 1-forms whose Euler-Lagrange equations are identically satisfied. This means that each m-sheet $x(t)$ could be an extremal. Such kind of Lagrange 1-form is called *null*. The definition of null Lagrange 1-forms does not depend on the coordinate system.

If a Lagrange 1-form can be written as $L_\alpha = D_\alpha L$, then

$$\int_{\Gamma_{t_0t_1}} L_\alpha dt^\alpha = \int_{\Gamma_{t_0t_1}} D_\alpha L\, dt^\alpha = L(t,x(t),x_\gamma(t))\Big|_{t_0}^{t_1}.$$

In this case, the functional depends only on the values of the function $x(t)$ taken on the boundary only and it is not affected by variations $h(t)$. Hence the 1-form $L_\alpha = D_\alpha L$ is a null one. Fortunately, these are all null Lagrange 1-forms.

Theorem 5.1. *A closed Lagrange 1-form $L_\alpha(t,x(t),x_\gamma(t))dt^\alpha$, $t\in\mathbb{R}^m_+$, is a null one if and only if it is a total divergence.*

Proof. The proof follows after we perform the same computations as we did for proving Theorem 4.2. □

Corollary 5.1. *The total divergence of a tensor of type $(1,1)$ is the total derivative of a function.*

Proposition 5.1. *If $L_\alpha dt^\alpha$ is a null Lagrange 1-form, then the Lagrange 1-forms*

$$\left(\frac{\partial L_\alpha}{\partial t^\beta}+D_\gamma\left(\frac{\partial x^i}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma}\right)+D_\mu\left(\frac{\partial L_\alpha}{\partial x^i_{\mu\nu}}\frac{\partial x^i_\nu}{\partial t^\beta}\right)-D_\mu\left(D_\nu\left(\frac{\partial L_\alpha}{\partial x^i_{\mu\nu}}\right)\frac{\partial x^i}{\partial t^\beta}\right)\right)dt^\alpha.$$

are exact.

Proof. The proof follows in the same manner as Proposition 4.1. □

We finish our considerations underlying that a wide class of optimization problems reduces to extremizing a *multi-time cost functional.* The multi-time cost functionals can be introduced either using a path independent curvilinear integral, or using a multiple integral. Using a similar argument as in [11], page 144, variational problems with multiple integrals can be converted to problems with curvilinear integrals and conversely.

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