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NONCONVEX QUASI VARIATIONAL INEQUALITIES

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

Abstract. In this paper, we introduce and consider a new class of quasi variational inequalities, which is called the *nonconvex quasi variational inequality.* We show that the projection method can be used to establish the equivalence between the nonconvex quasi variational inequalities and the fixed points problems and the Wiener-Hopf equations. We use this equivalent alternative formulation to suggest and analyze a new class of two-step iterative methods for solving the nonconvex variational inequalities. We discuss the convergence of the iterative method under suitable conditions conditions. Our method of proofs is very simple as compared with other techniques.

1. INTRODUCTION

Quasi variational inequalities are being used to study a wide class of problems with applications in various branches of pure, applied, engineering and medical sciences, the origin of which can be traced back to Fermat, Newton, Leibniz, Bernouli, Euler and Lagrange. Quasi variational inequalities can be viewed as a natural and useful generalization of the variational inequalities which were introduced by Stampacchia [42]. It is well-known that if the underlying convex set in the formulation of the variational inequality also depends upon the solution explicitly or implicitly, then the variational inequalities is called the quasi variational inequalities, see Bensoussan and Lions [2]. In recent years, considerable attention has been given to develop several numerical techniques for solving variational inequalities. There is a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle, and descent framework for solving variational inequalities and complementarity problems; see [1]-[42] and the references therein. It is worth mentioning that almost all the results regarding the existence and iterative schemes for variational inequalities, which have been investigated and considered in the classical convexity. This is because all the techniques

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are based on the properties of the projection operator over convex sets, which may not hold in general for the nonconvex sets.

Noor [26], [29] has introduced and studied a new class of variational inequalities, which is called the nonconvex variational inequality in conjunction with the uniformly prox-regular sets. It is well-known that the uniformly prox-regular sets are nonconvex sets and include the convex sets as a special case, see [8], [35]. Noor, [26], [29] has shown that the projection technique can be extended for the nonconvex variational inequalities.

Inspired and motivated by the ongoing research in this area, we introduce and study the nonconvex quasi variational inequalities. Using essentially the technique of Noor [30], [31], [35], we establish the equivalence between the nonconvex quasi variational inequalities and fixed-point problems. This equivalent alternative formulation is used to discuss the existence of a solution of the nonconvex variational inequalities, which is Theorem 3.1. We use this alternative equivalent formulation to suggest and analyze an implicit type iterative methods for solving the nonconvex variational inequalities. In order to implement this new implicit method, we use the predictor-corrector technique to suggest a two-step method for solving the nonconvex variational inequalities, which is Algorithm 3.5. We also consider the convergence (Theorem 3.2) of the new iterative method under some suitable conditions. We have also suggested three-step iterative methods for solving nonconvex variational inequalities. Some special cases are also discussed.

We also introduce and consider the problem of solving the implicit Wiener-Hopf equations. Using essentially the projection technique, we show that the nonconvex variational inequalities are equivalent to the Wiener-Hopf equations. This alternative equivalent formulation is more general and flexible than the projection operator technique. This alternative equivalent formulation is used to suggest and analyze a number of iterative methods for solving the nonconvex variational inequalities. These iterative methods is the subject of Section 4. We also consider the convergence criteria of the proposed iterative methods under some suitable conditions. Several special cases are also discussed. Results obtained in this paper can be viewed as refinement and improvement of the previously known results for the variational inequalities and related optimization problems. We would like to point out that our method of proofs is very simple as compared with other techniques.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [8], [39].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(\cdot)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. *Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall\ v \in K.$$

Poliquin et al. [39] and Clarke et al. [8] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.2. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall\ u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u \rangle \leq \frac{1}{2r} \|v - u\|^2, \quad \forall\ v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [8], [39]. It is clear that if $r = \infty$ and $K_r(u) \equiv K$, then uniformly prox-regularity of $K_r(u)$ is equivalent to the convexity of K. It is known that if $K_r(u)$ is a uniformly prox-regular set, then the proximal normal cone $N_{K_r}^P(u)$ is closed as a set-valued mapping.

For a given nonlinear operator T and a point-to-set mapping $K_r : u \longrightarrow K_r(u)$, which associates a closed uniformly prox-regular set $K_r(u)$ of H with any element of H, we consider the problem of finding $u \in K_r(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K_r(u), \tag{2.1}$$

which is called the *nonconvex quasi variational inequality.*

We note that, if $K_r(u) \equiv K_r$, the uniformly prox-regular set in H, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \forall v \in K_r. \tag{2.2}$$

Inequality of type (2.2) is called the *nonconvex variational inequality*, which was considered and studied by Noor [30], [31], [35] recently.

If $K_r(u) \equiv K(u)$, then problem (2.1) reduces to: find $u \in K(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u), \tag{2.3}$$

which is called the *quasi variational inequality,* introduced and studied by Bensoussan and Lions [2]. For the applications, numerical methods, physical formulations and other aspects of quasi variational inequalities, see [2]-[42] and the references therein.

If $K_r(u) \equiv K$, the convex set, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{2.4}$$

which is called the *variational inequality* introduced and studied by Stampacchia [42] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [2]-[42] and the references therein.

It is well-known that problem (2.2) is equivalent to finding $u \in K$ such that

$$0 \in Tu + N_K(u), \tag{2.5}$$

where $N_K(u)$ denotes the normal cone of K at u in the sense of convex analysis. Problem (2.5) is called the variational inclusion associated with variational inequality (2.4).

Similarly, if $K_r(u)$ is a nonconvex (uniformly prox-regular) set, then problem (2.1) is equivalent to finding $u \in K_r(u)$ such that

$$0 \in Tu + N^P_{K_r(u)}(u), \tag{2.6}$$

where $N^P_{K_r(u)}(u)$ denotes the normal cone of $K_r(u)$ at u in the sense of nonconvex analysis. Problem (2.6) is called the quasi nonconvex variational inclusion problem associated with nonconvex quasi variational inequality (2.1). This implies that the variational inequality (2.1) is equivalent to finding a zero of the sum of two monotone operators (2.6). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the nonconvex quasi variational inequality (2.1).

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets.

Lemma 2.2. *Let K be a nonempty closed subset of H, $r \in (0,\infty]$ and set $K_r = \{u \in H : d(u,K) < r\}$. If K_r is uniformly prox-regular, then*

i) $\forall u \in K_r, P_{K_r}(u) \neq \emptyset$.

ii) $\forall r' \in (0,r)$, P_{K_r} *is Lipschitz continuous with constant* $\dfrac{r}{r-r'}$ *on* $K_{r'}$.

iii) *The proximal normal cone is closed as a set-valued mapping.*

We now consider the problem of solving the nonlinear Wiener-Hopf equations. To be more precise, let $Q_{K_r(u)} = I - P_{K_r(u)}$, where $P_{K_r(u)}$ is the projection operator and I is the identity operator. For a given nonlinear operator T, consider the problem of finding $z \in H$ such that

$$TP_{K_r(u)}z + \rho^{-1}Q_{K_r(u)}z = 0. \tag{2.7}$$

Equations of the type (2.7) are called the nonconvex implicit Wiener-Hopf equations. Note that if $r = \infty$ and $K_r(u) \equiv K$, the convex set, then the nonlinear implicit Wiener-Hopf equations are exactly the same Wiener-Hopf equations associated with the variational inequalities (2.2), which were introduced and studied by Shi [40]. This shows that the original Wiener-Hopf equations are the special case of the nonlinear Wiener-Hopf equations (2.7). The Wiener-Hopf equations technique has been used to study and develop several iterative methods for solving variational inequalities and related optimization problems, see [10]-[27].

Definition 2.3. An operator $T : H \to H$ is said to be:

i) *strongly monotone*, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2, \quad \forall u, v \in H.$$

ii) *Lipschitz continuous*, if and only if, there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||, \quad \forall u, v \in H.$$

We would like to point out that the implicit projection operator $P_{K_r(u)}$ is not nonexpansive. We shall assume that the implicit projection operator $P_{K_r(u)}$ satisfies the Lipschitz type continuity, which plays an important and fundamental role in the existence theory and in developing numerical methods for solving quasi variational inequalities.

Assumption 2.1 For all $u, v, w \in H$, the implicit projection operator $P_{K_r(u)}$ satisfies the condition

$$\|P_{K_r(u)}w - P_{K_r(v)}w\| \leq \nu\|u - v||, \tag{2.8}$$

where $\nu > 0$ is a positive constant.

3. PROJECTION METHODS

In this section, we establish the equivalence between the nonconvex quasi variational inequality (2.1) and the fixed point problem using the projection operator technique. This alternative formulation is used to discuss the existence of a solution of the problem (2.1) and to suggest some new iterative methods for solving the nonconvex quasi variational inequality (2.1).

Lemma 3.1. *$u \in K_r(u)$ is a solution of the nonconvex quasi variational inequality* (2.1) *if and only if $u \in K_r(u)$ satisfies the relation*

$$u = P_{K_r(u)}[u - \rho Tu], \tag{3.1}$$

where $P_{K_r(u)}$ is the projection of H onto the uniformly prox-regular set $K_r(u)$.

Proof. Let $u \in K_r(u)$ be a solution of (2.1). Then, for a constant $\rho > 0$,

$$\begin{aligned} 0 &\in u + \rho N^P_{K_r(u)}(u) - (u - \rho Tu) = (I + \rho N^P_{K_r(u)})(u) - (u - \rho Tu) \\ &\Leftrightarrow \\ u &= (I + \rho N^P_{K_r(u)})^{-1}[u - \rho Tu] = P_{K_r(u)}[u - \rho Tu], \end{aligned}$$

where we have used the well-known fact that $P_{K_r(u)} \equiv (I + N^P_{K_r(u)})^{-1}$. □

Lemma 3.1 implies that the nonconvex quasi variational inequality (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical point of views.

We rewrite the the relation (3.1) in the following form

$$F(u) = P_{K_r(u)}[u - \rho Tu], \tag{3.2}$$

which is used to study the existence of a solution of the nonconvex variational inequality (2.1).

We now study those conditions under which the nonconvex variational inequality (2.1) has a solution and this is the main motivation of our next result, which is due to Noor [26], [29].

Theorem 3.1. *Let $P_{K_r(u)}$ be the Lipschitz continuous operator with constant $\delta = \frac{r}{r - r'}$ and satisfy the Assumption* 2.1 *with constant $\nu > 0$. Let T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. If there exists a constant ρ such that*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\delta^2\alpha^2 - \beta^2(\delta^2 - (1-\nu)^2)}}{\delta\beta^2}, \tag{3.3}$$

$$\delta\alpha > \beta\sqrt{\delta^2 - (1-\nu)^2}, \quad \nu < 1, \delta > 1, \tag{3.4}$$

then there exists a solution of the problem (2.1).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2), has a fixed point. For all $u \neq v \in K_r(u)$, we have

$$\begin{aligned} ||F(u) - F(v)|| &= \|P_{K_r(u)}[u - \rho Tu] - P_{K_r(v)}[v - \rho Tv]|| \\ &\leq \|P_{K_r(u)}[u - \rho Tu] - \|P_{K_r(v)}[u - \rho Tu]\| \\ &\quad + \|P_{K_r(v)}[u - \rho Tu]\|P_{K_r(v)}[v - \rho Tv]\| \\ &\leq \nu\|u - v\| + \delta\|u - v - \rho(Tu - Tv)\|, \end{aligned} \tag{3.5}$$

where we have used the fact that the operator P_{K_r} is a Lipschitz continuous operator with constant δ and the Assumption 2.1.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned}||u - v - \rho(Tu - Tv)||^2 &\leq ||u - v||^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2||Tu - Tv||^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)||u - v||^2. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have

$$\|F(u) - F(v)\| \leq \left\{\nu + \delta\sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\right\} \|u - v\| = \theta||u - v||,$$

where

$$\theta = \left\{\nu + \delta\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right\}. \tag{3.7}$$

From (3.3) and (3.4), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (3.2), has a fixed point, which is the unique solution of (2.1). □

This fixed point formulation (3.1) is used to suggest the following iterative method for solving the nonconvex variational inequality (2.1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K_r(u_n)}[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

If $K_r(u) \equiv K_r$, then Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K_r}[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

For the convergence analysis of Algorithm 3.2, see Noor [26], [29].

We again use the fixed formulation to suggest and analyze an iterative method for solving the nonconvex quasi variational inequalities (2.1) as:

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K_r(u_n)}[u_{n+1} - \rho Tu_{n+1}], \quad n = 0, 1, 2, \ldots$$

Algorithm 3.3 is an implicit type iterative method, which is difficult to implement. To implement Algorithm 3.3, we use the predictor-corrector technique. Here we use the Algorithm 3.1 as a predictor and Algorithm 3.3 as a corrector. Consequently, we have the following iterative method

Algorithm 3.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= P_{K_r(u_n)}[u_n - \rho Tu_n] \\ u_{n+1} &= P_{K_r(y_n)}[y_n - \rho Ty_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is called the two-step or splitting type iterative method for solving the nonconvex variational inequalities (2.1). It is worth mentioning that Algorithm 3.3 can be suggested by using the updating the technique of the solution.

In this paper, we suggest and analyze the following two-step iterative method for solving the nonconvex variational inequalities (2.1).

Algorithm 3.5. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$y_n = (1-\beta_n)x_n + \beta_n P_{K_r(u_n)}[u_n - \rho T u_n] \tag{3.8}$$

$$u_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{K_r(y_n)}[y_n - \rho T y_n], \quad n = 0, 1, 2, \ldots, \tag{3.9}$$

where $\alpha_n, \beta_n \in [0,1], \quad \forall n \geq 0$.

Clearly for $\alpha_n = \beta_n = 1$, Algorithm 3.5 reduces to Algorithm 3.4.

It is worth mentioning that if $r = \infty$, then the nonconvex set set $K_r(u)$ reduces to a convex set $K(u)$. Consequently Algorithms 3.1- 3.5 collapse to the following algorithms for solving the classical quasi variational inequalities (2.2). We would like to point that Algorithm 3.3 appears to be a new one for solving the variational inequalities (2.2)

We now consider the convergence analysis of Algorithm 3.5 and this is the main motivation of our next result. Similarly, one can consider the convergence analysis of other algorithms.

Theorem 3.2. *Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \dfrac{r}{r-r'}$ and Assumption 2.1 hold. Let the operator $T\colon H \longrightarrow H$ be strongly monotone with constants $\alpha > 0$ and Lipschitz continuous with constants with $\beta > 0$. If (3.3) holds, $\alpha_n, \beta_n \in [0,1], \quad \forall n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.5 converges to a solution $u \in K_r$ satisfying the nonconvex variational inequality (2.1).*

Proof. Let $u \in K_r(u)$ be a solution of the nonconvex variational inequality (2.1). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n P_{K_r(u)}[u - \rho T u], \tag{3.10}$$

$$= (1-\beta_n)u + \beta_n P_{K_r(u)}[u - \rho T u], \tag{3.11}$$

where $0 \leq \alpha_n, \beta_n \leq 1$ are constants.

From (3.5), (3.7), (3.9), (3.10), using the Lipschitz continuity of the projection P_{K_r} with constant $\delta > 0$ and Assumption 2.1, we have

$$\begin{aligned}||u_{n+1} - u|| &= ||(1-\alpha_n)(u_n - u) + \alpha_n\{P_{K_r(y_n)}[y_n - \rho Ty_n] - P_{K_r(u)}[u - \rho Tu]\}|| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\|P_{K_r}[y_n - \rho Ty_n] - P_{K_r}[u - \rho Tu]\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\delta\|y_n - u + \rho(Ty_n - Tu)\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\delta\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\|y_n - u\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|y_n - u\|. \end{aligned} \tag{3.12}$$

From (3.8), (3.6), (3.7) and (3.11), we have

$$\begin{aligned}\|y_n - u\| &\leq (1-\beta_n)\|u_n - u\| + \beta_n\theta\|u_n - u\| \\ &= \{1 - \beta_n(1-\theta)\}\,\|u_n - u\|. \end{aligned} \tag{3.13}$$

Combining (3.13) and (3.12), we have

$$\begin{aligned}\|u_{n+1} - u\| &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\{1 - \beta_n(1-\theta)))\}\|u_n - u\| \\ &= [1 - \alpha_n(1 - \theta((1 - \beta_n(1-\theta)))]\,\|u_n - u\| \\ &\leq \prod_{i=0}^{n}[1 - \alpha_i(1-\theta_1)]\,\|u_0 - u\|, \end{aligned}$$

where

$$\theta_1 = \theta((1 - \beta_n(1-\theta)) < 1.$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1 - \theta_1 > 0$, we have $\lim_{n\to\infty}\left\{\prod_{i=0}^{n}[1 - (1-\theta_1)\alpha_i]\right\} = 0.$ Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

4. WIENER-HOPF EQUATIONS TECHNIQUE

In this section, we first establish the equivalence between the nonconvex quasi variational inequality (2.1) and the Wiener-Hopf equations (2.7) using essentially the projection method. This equivalence is used to suggest and analyze some iterative methods for solving the nonconvex quasi variational inequalities (2.1).

Using Lemma 3.1, we show that the nonconvex quasi variational inequalities (2.1) are equivalent to the Wiener-Hopf equations (2.7).

Lemma 4.1. *The nonconvex quasi variational inequality* (2.1) *has a solution* $u \in K_r(u)$ *if and only if the Wiener-Hopf equations* (2.7) *have a solution* $z \in H$, *provided*

$$u = P_{K_r(u)}z \tag{4.1}$$

$$z = u - \rho Tu, \tag{4.2}$$

where $\rho > 0$ *is a constant.*

Proof. Let $u \in K_r(u)$ be a solution of (2.1). Then, from Lemma 3.1, we have

$$u = P_{K_r(u)}[u - \rho Tu]. \tag{4.3}$$

Taking $z = u - \rho Tu$ in (4.3), we have

$$u = P_{K_r(u)}z. \tag{4.4}$$

From (4.3) and (4.4), we have

$$z = u - \rho Tu = P_{K_r(u)}z - \rho TP_{K_r}z,$$

which shows that $z \in H$ is a solution of the Wiener-Hopf equations (2.7). This completes the proof. □

From Lemma 4.1, we conclude that the variational inequality (2.1) and the Wiener-Hopf equations (2.7) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving variational inequalities and related optimization problems. In this paper, by suitable and appropriate rearrangement, we suggest a number of new iterative methods for solving the nonconvex variational inequality (2.1).

I. The Wiener-Hopf equations (2.7) can be written as

$$P_{K_r(u)}z = -\rho TP_{K_r(u)}z,$$

which implies that, using (4.2)

$$z = P_{K_r(u)}z - \rho TP_{K_r(u)}z = u - \rho Tu.$$

This fixed point formulation enables us to suggest the following iterative method for solving the nonconvex variational inequality (2.1).

Algorithm 4.1. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$u_n = P_{K_r(u_n)}z_n \tag{4.5}$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{u_n - \rho Tu_n,\} \quad n = 0, 1, 2, \dots, \tag{4.6}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. The Wiener-Hopf equations (2.7) may be written as

$$\begin{aligned} z &= P_{K_r(u)}z - \rho TP_{K_r(u)}z + (1 - \rho^{-1})Q_{K_r}z \\ &= u - \rho Tu + (1 - \rho^{-1})Q_{K_r(u)}z. \end{aligned}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$u_n = P_{K_r(u_n)}z_n$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{u_n - \rho Tu_n + (1 - \rho^{-1})Q_{K_r(u_n)}z_n,\} \quad n = 0, 1, 2, \dots,$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

III. If the operator T is linear and T^{-1} exists, then the Wiener-Hopf equation (2.7) can be written as

$$z = (I - \rho^{-1}T^{-1})Q_{K_r(u)}z,$$

which allows us to suggest the iterative method.

Algorithm 4.3. For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{(I - \rho^{-1}T^{-1})Q_{K_r(u_n)}z_n,\} \quad n = 0, 1, 2 \ldots,$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We would like to point out that one can obtain a number of iterative methods for solving the variational inequality (2.1) for suitable and appropriate choices of the operators T, A and the space H. This shows that iterative methods suggested in this paper are more general and unifying ones.

We now study the convergence analysis of Algorithm 4.1. In a similar way, one can analyze the convergence analysis of other iterative methods.

Theorem 4.1. *Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \dfrac{r}{r - r'}$ and Assumption 2.1 hold. Let the operator $T\colon H \to H$ be strongly monotone with constants $\alpha > 0$ and Lipschitz continuous with constants with $\beta > 0$. If the condition (3.3) holds, $\alpha_n \in [0, 1]$, $\forall n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution $\{z_n\}$ obtained from Algorithm 4.1 converges to a solution $z \in H$ satisfying the Wiener-Hopf equation (2.7) strongly.*

Proof. Let $u \in K_r(u)$ be a solution of (2.1). Then, using Lemma 4.1, we have

$$z = (1 - \alpha_n)z + \alpha_n\{u - \rho Tu\}, \tag{4.7}$$

where $0 \leq \alpha_n \leq 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From (4.7), (4.6) and (3.5), we have

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\|u_n - u - \rho(Tu_n - Tu)\| \\ &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\left\{\sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\right\}\|u_n - u\|. \end{aligned} \tag{4.8}$$

Also from (4.5), (4.1) and the Lipschitz continuity of the projection operator P_{K_r} with constant δ, we have

$$\begin{aligned}\|u_n - u\| &= \|P_{K_r(u_n)}z_n - P_{K_r(u)}z\| \\ &\leq \|P_{K_r(u_n)}z_n - P_{K_r(u)}z_n\| + \|P_{K_r(u)}z_n - P_{K_r(u)}z\| \\ &\leq \nu\|u_n - u\| + \delta\|z_n - z\|,\end{aligned}$$

from which, we have

$$\|u_n - u\| \leq \frac{1}{1-\nu}\|z_n - z\|. \tag{4.9}$$

Combining (4.8), and (4.9), we have

$$\begin{aligned}\|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\delta\left\{\frac{\sqrt{1-2\rho\alpha+\beta^2\rho^2}}{1-\nu}\right\}\|z_n - z\| \\ &= (1-\alpha_n)\|z_n - z\| + \alpha_n\theta_2\|z_n - z\|, \end{aligned} \tag{4.10}$$

where $\theta_2 = \delta\dfrac{\sqrt{1-2\rho\alpha+\rho^2\beta^2}}{1-\nu}$.

From (3.3) and (3.4), we see that $\theta_2 < 1$ and consequently

$$\begin{aligned}||z_{n+1} - z|| &\leq (1-\alpha_n)||z_n - z|| + \alpha_n\theta_2||z_n - z|| \\ &= [1-(1-\theta_2)\alpha_n]||z_n - z|| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta_2)\alpha_i]||z_0 - z||.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta_2 > 0$, we have $\lim_{n\to\infty}\prod_{i=0}^{n}[1-(1-\theta)\alpha_i] = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H, the required result. □

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