J. Adv. Math. Studies
Vol. **3**(2010), No. 1, 41-58
http://journal.fairpartners.ro

MOND-WEIR DUALITIES WITH LAGRANGIANS FOR MULTIOBJECTIVE FRACTIONAL AND NON-FRACTIONAL VARIATIONAL PROBLEMS

ŞTEFAN MITITELU AND MIHAI POSTOLACHE

Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

ABSTRACT. Based on the normal efficiency conditions for a multiobjective non-fractional variational problem, we consider fractional duals of generalized Mond-Weir type and fractional Mond-Weir-Zalmai type. Under some assumptions of (ρ, b)-quasiinvexity, weak, direct and converse duality theorems are stated in various variants.

INTRODUCTION

The first result on the necessity of optimal solutions of scalar variational problems was established by Valentine [19] in 1937. The papers of Mond and Hanson [10], Mond, Chandra and Husain [11], Mond and Husain [12], Preda [16] developed the duality of the scalar variational problems involving convex and generalized convex functions. Mititelu [3], Mititelu and Stancu-Minasian [9], Mukherjee and Purnachandra [13], established weak efficiency conditions and developed different types of dualities for multiobjective variational problems under various types of generalized convex functions. Kim and Kim [2] used the efficiency property of the nondifferentiable multiobjective variational problems in duality theory.

In this paper, we use the notion of normal efficient solution introduced by Mititelu [5] and establish certain new results of Mond-Weir duality type and of Zalmai duality type for multiobjective fractional and non-fractional variational problems using (ρ, b)-quasiinvexity assumptions.

For related but different results obtained by other authors on this topic, we address the reader to: [1] by R. Jagannathan, [14] and [15] by Ariana Pitea, C. Udrişte and Şt. Mititelu, [18] by I. M. Stancu-Minasian and Şt. Mititelu.

1. NOTATIONS AND PROBLEMS STATEMENT

In $\mathbb{R}^n$, the n-dimensional Euclidean space, consider the vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. We recall that the relations $v = w$, $v < w$, $v \leqq w$, $v \leq w$ are

Received: May 1, 2009.
2000 *Mathematics Subject Classification*: 65K10, 90C29, 26B25.
Key words and phrases: multiobjective fractional problem, efficient solution, quasiinvexity.

defined as follows:

$$v = w \Leftrightarrow v_i = w_i, \quad i = \overline{1,n}; \qquad v < w \Leftrightarrow v_i < w_i, \quad i = \overline{1,n};$$
$$v \leqq w \Leftrightarrow v_i \leqq w_i, \quad i = \overline{1,n}; \qquad v \leq w \Leftrightarrow v \leqq w \text{ and } v \neq w.$$

Let $I = [a,b]$ be a real interval and $f = (f_1,\dots,f_p)\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $k = (k_1,\dots,k_p)\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $g = (g_1,\dots,g_m)\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, $h = (h_1,\dots,h_q)\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$ be two times differentiable functions.

Consider a vector-valued function $f(t,x,\dot{x})$, where $t \in I$, $x\colon I \to \mathbb{R}^n$ and $\dot{x} = \dfrac{\mathrm{d}x}{\mathrm{d}t}$. Denote by f_x and $f_{\dot{x}}$ the $p \times n$ matrices of first-order partial derivatives with respect to x and $\dot{x}$ respectively, that is $f_x = (f_{1x}, f_{2x}, \dots, f_{px})'$ and $f_{\dot{x}} = (f_{1\dot{x}}, f_{2\dot{x}}, \dots, f_{p\dot{x}})'$, with

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \dots, \frac{\partial f_i}{\partial x_n}\right) \quad \text{and} \quad f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \dots, \frac{\partial f_i}{\partial \dot{x}_n}\right), \quad i = 1,2,\dots,p.$$

By analogy, k_x, g_x, h_x and $k_{\dot{x}}$, $g_{\dot{x}}$, $h_{\dot{x}}$ denote the $p \times n$, $n \times n$, $q \times n$ matrices of the first order partial derivatives of k, g and h respectively, with respect to x and $\dot{x}$.

Let X denote the space of piecewise smooth (continuously differentiable) functions x with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$ is given by $u = Dx \Leftrightarrow x(t) = x(a) + \displaystyle\int_a^t u(s)\mathrm{d}s$, excepting the discontinuities, where $x(a)$ is a given boundary value.

Consider the multiobjective fractional variational problem

$$(\text{MFP}) \begin{cases} \text{Minimize } \left(\dfrac{\displaystyle\int_a^b f_1(t,x(t),\dot{x}(t))\mathrm{d}t}{\displaystyle\int_a^b k_1(t,x(t),\dot{x}(t))\mathrm{d}t}, \dots, \dfrac{\displaystyle\int_a^b f_p(t,x(t),\dot{x}(t))\mathrm{d}t}{\displaystyle\int_a^b k_p(t,x(t),\dot{x}(t))\mathrm{d}t} \right) \\ \text{subject to} \\ \quad x(a) = a_0, \quad x(b) = b_0, \\ \quad g(t,x(t),\dot{x}(t)) \leqq 0, \quad h(t,x(t),\dot{x}(t)) = 0, \quad \forall t \in I. \end{cases}$$

Assume that $\displaystyle\int_a^b k_i(t,x(t),\dot{x}(t))\mathrm{d}t > 0$ for all $i = 1,2,\dots,p$. Let

$$\mathcal{D} = \{x \in X \mid x(a) = a_0,\ x(b) = b_0,\ g(t,x(t),\dot{x}(t)) \leqq 0,\ h(t,x(t),\dot{x}(t)) = 0,\ \forall t \in I\}$$

be the set of all feasible solutions of problem (MFP).

Consider also the following multiobjective variational problem

$$(\text{MP}) \begin{cases} \min \displaystyle\int_a^b f(t,x(t),\dot{x}(t))\mathrm{d}t = \left(\int_a^b f_1(t,x(t),\dot{x}(t))\mathrm{d}t, \dots, \int_a^b f_p(t,x(t),\dot{x}(t))\mathrm{d}t \right) \\ \text{subject to} \\ \quad x(a) = a_0, \quad x(b) = b_0, \\ \quad g(t,x(t),\dot{x}(t)) \leqq 0, \quad h(t,x(t),\dot{x}(t)) = 0, \quad t \in I. \end{cases}$$

and remark that the domain of (MP) is also $\mathcal{D}$.

In this paper, we define a duality of Mond-Weir-Zalmai type for problem (MFP) through weak, direct and converse duality theorems. For problem (MP) is developed a duality of generalized Mond-Weir type, also giving theorems of the same type. We give various variants of these results using assumptions of (ρ, b)-quasiinvexity.

2. PRELIMINARIES ON OPTIMALITY AND EFFICIENCY FOR VARIATIONAL PROBLEMS

In this section we recall some definitions and auxiliary results that will be needed later in our discussion of efficiency conditions and Mond-Weir duality for problem (MFP).

Definition 2.1. A feasible solution $x^0 \in \mathcal{D}$ is an *efficient solution* of problem (MP) if there is no $x \in \mathcal{D}$, $x \neq x^0$, such that

$$\int_a^b f(t, x(t), \dot{x}(t))\mathrm{d}t \leq \int_a^b f(t, x^0(t), \dot{x}^0(t))\mathrm{d}t.$$

For problem (MP) we quote the following result of efficiency

Theorem 2.1 (Necessary efficiency conditions for problem (MP))**.** ([6], [9]) *Let $x^0 \in \mathcal{D}$ be an efficient solution of problem* (MP)*. Then there exist a vector $\lambda^0 \in \mathbb{R}^p$ and piecewise smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ which satisfy the following conditions*

$$(\mathrm{MV}) \begin{cases} \lambda^{0\,\prime} f_x(t, x^0(t), \dot{x}^0(t)) + \mu^0(t)' g_x(t, x^0(t), \dot{x}^0(t)) + \nu^0(t)' h_x(t, x^0(t), \dot{x}^0(t)) \\ \quad = \dfrac{\mathrm{d}}{\mathrm{d}t}\big[\lambda^{0\,\prime} f_{\dot{x}}(t, x^0(t), \dot{x}^0(t)) + \mu^0(t)' g_{\dot{x}}(t, x^0(t), \dot{x}^0(t)) \\ \quad\quad + \nu^0(t)' h_{\dot{x}}(t, x^0(t), \dot{x}^0(t))\big] \\ \mu^0(t)' g(t, x^0(t), \dot{x}^0(t)) = 0, \quad \mu_i(t) \geqq 0, \quad \forall t \in I, \\ \lambda^0 \geqq 0,\ e'\lambda^0 = 1, \quad e = (1, \ldots, 1)' \in \mathbb{R}^p \text{ if } \lambda^0 \geq 0. \end{cases}$$

Definition 2.2. [5] The point $x^0 \in \mathcal{D}$ is a *normal efficient solution* of problem (MP) if the vector λ^0 from the previous theorem satisfies the inequality $\lambda^0 \geq 0$.

We present now the efficiency necessary conditions for problem (MFP).

For each $i = \overline{1, p}$, we consider the functionals

$$F_i, K_i \colon X \to \mathbb{R}, \quad F_i(x) = \int_a^b f_i(t, x(t), \dot{x}(t))\mathrm{d}t, \quad K_i(x) = \int_a^b k_i(t, x(t), \dot{x}(t))\mathrm{d}t.$$

Mititelu established the following necessary efficiency conditions for (MFP):

Theorem 2.2 (Necessary efficiency conditions for problem (MFP))**.** ([6], [9]). *Let x^0 be a normal efficient solution of problem* (MFP)*. Then there exist $\lambda^0 \in \mathbb{R}^n$ and* *piecewise smooth functions* $\mu^0\colon I\to\mathbb{R}^q$ *and* $\nu^0\colon I\to\mathbb{R}^n$ *that satisfy the conditions*

$$(\mathrm{MFV})\begin{cases}\displaystyle\sum_{i=1}^{p}\lambda_i^0\big[K_i(x^0)f_{ix}(t,x^0(t),\dot{x}^0(t))-F_i(x^0)k_{ix}(t,x^0(t),\dot{x}^0(t))\big]\\ \quad+\mu^0(t)'g_x(t,x^0(t),\dot{x}^0(t))+\nu^0(t)'h_x(t,x^0(t),\dot{x}^0(t))\\ \quad\displaystyle=\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{i=1}^{p}\lambda_i^0\big[K_i(x^0)f_{i\dot{x}}(t,x^0(t),\dot{x}^0(t))-F_i(x^0)k_{i\dot{x}}(t,x^0(t),\dot{x}^0(t))\big]\\ \quad+\mu^0(t)'g_{\dot{x}}(t,x^0(t),\dot{x}^0(t))+\nu^0(t)'h_{\dot{x}}(t,x^0(t),\dot{x}^0(t))\Big\}\\ \mu^0(t)'g(t,x^0(t),\dot{x}^0(t))=0,\quad \mu^0(t)\geqq 0,\quad \forall t\in I,\\ \lambda^0\geq 0,\quad e'\lambda^0=1.\end{cases}$$

Let $\rho\in\mathbb{R}^n$ and $b\colon X\times X\to[0,\infty)$. Denote $H(x)=\displaystyle\int_a^b h(t,x(t),\dot{x}(t))\mathrm{d}t$.

Definition 2.3. ([6]) The function H is called *(strictly)* (ρ,b)*-quasiinvex* at the point x^0 if there exist vector functions $\eta\colon I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^q$, with $\eta(t,x(t),\dot{x}(t))=0$ for $x(t)=x^0(t)$, and $\theta\colon X\times X\to\mathbb{R}^n$ such that for any x $(x\neq x^0)$,

$$H(x)\leqq H(x^0)\ \Rightarrow\ b(x,x^0)\int_a^b\big[\eta' h_x(t,x^0(t),\dot{x}^0(t))+(D\eta)'h_{\dot{x}}(t,x^0(t),\dot{x}^0(t))\big]\mathrm{d}t$$
$$(<)\leqq-\rho b(x,x^0)\big\|\theta(x,x^0)\big\|^2.$$

Definition 2.4. ([3]) The function H is *monotonic* (ρ,b)*-quasiinvex* at the point x^0 if there exist vector functions $\eta\colon I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^q$ with $\eta(t,x(t),\dot{x}(t))=0$ for $x(t)=x^0(t)$, and $\theta\colon X\times X\to\mathbb{R}^n$ such that for any x $(x\neq x^0)$,

$$H(x)=H(x^0)\ \Rightarrow\ b(x,x^0)\int_a^b\big[\eta' h_x(t,x^0(t),\dot{x}^0(t))+(D\eta)'h_{\dot{x}}(t,x^0(t),\dot{x}^0(t))\big]\mathrm{d}t$$
$$=-\rho b(x,x^0)\big\|\theta(x,x^0)\big\|^2.$$

3. GENERALIZED MOND-WEIR DUALITY FOR (MP)

Let $\{J_0,J_1,\dots,J_r\}$ be a partition of the set $J=\{1,\dots,m\}$ that is $J_\alpha\subseteq J$, $J_\alpha\cap J_\beta=\emptyset$ if $\alpha\neq\beta$, $\displaystyle\bigcup_{\alpha=0}^{r}J_\alpha=J$. Let also $\{S_0,S_1,\dots,S_r\}$ be a partition of the set $S=\{1,\dots,q\}$. Consider a function $y\in X$ and the piecewise non-smooth functions $\mu\colon I\to\mathbb{R}^m$ and $\nu\colon I\to\mathbb{R}^q$. The Mond-Weir Lagrangian associated to problem (MP) is

$$L_e(t,y(t),\dot{y}(t))\ =\ f(t,y(t),\dot{y}(t))$$
$$+\big[\mu_{J_0}(t)'g_{J_0}(t,y(t),\dot{y}(t))+\nu_{S_0}(t)'h_{S_0}(t,y(t),\dot{y}(t))\big]\,e,$$

where $L_e = (L_1, \dots, L_p)$ and

$$L_i(t, y(t), \dot{y}(t)) = f_i(t, y(t), \dot{y}(t)) + \mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t)),$$

for $i = \overline{1,p}$.

The generalized multiobjective Mond-Weir dual problem associated to problem (MP) is the following variational problem:

$$(\text{MWD}) \begin{cases} \text{Maximize (Pareto)} \displaystyle\int_a^b L_e(t, y(t), \dot{y}(t)) \mathrm{d}t \\ \quad = \left(\displaystyle\int_a^b L_1(t, y(t), \dot{y}(t)), \dots, \int_a^b L_p(t, y(t), \dot{y}(t)) \mathrm{d}t \right) \\ \text{subject to} \\ \quad y(a) = y_0, \quad y(b) = b_0, \\ \quad \lambda' f_x(t, y(t), \dot{y}(t)) + \mu(t)' g_x(t, y(t), \dot{y}(t)) + \nu(t)' h_x(t, y(t), \dot{y}(t)) \\ \qquad = \dfrac{\mathrm{d}}{\mathrm{d}t} \left\{ \lambda' f_{\dot{x}}(t, y(t), \dot{y}(t)) + \mu(t)' g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)' h_{\dot{x}}(t, y(t), \dot{y}(t)) \right\} \\ \quad \mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t, y(t), \dot{y}(t)) \geqq 0, \ \alpha = \overline{1,r}, \ \forall t \in I, \\ \quad \lambda \geq 0, \quad e' \lambda = 1. \end{cases}$$

We denote by $\pi(x)$ the value of problem (MP) at $x \in \mathcal{D}$ and by $\delta(y, \lambda, \mu, \nu)$ the value of dual (MWD) at $(y, \lambda, \mu, \nu) \in \Delta$, where Δ is the domain of (MWD).

• CASE STUDY 1

Theorem 3.1 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems* (MP) *and* (MWD). *Assume that the following conditions are satisfied:*

a) *For each $i = \overline{1,p}$, the integral $\displaystyle\int_a^b L_i(t, x(t), \mu(t), \nu(t)) \mathrm{d}t$ is (ρ_i', b)-quasiinvex at the point y with respect to η and θ.*

b) *For each $\alpha = \overline{1,r}$, $\displaystyle\int_a^b \left[\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t)) \right] \mathrm{d}t$ is (ρ_α'', b)-quasiinvex at the point y with respect to η and θ.*

c) *At least one of the integrals from* a)-b) *is strictly (ρ, b)-quasiinvex at the point y with respect to η and θ.*

d) $\displaystyle\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha'' \geqq 0.$

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Proof. By reductio ad absurdum, suppose there exist $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ such that $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$. Hence, for all $i = \overline{1,p}$, we have

$$\int_a^b f(t, x(t), \dot{x}(t)) \mathrm{d}t \leq \int_a^b L_e(t, y(t), \dot{y}(t)) \mathrm{d}t,$$

or componentwise

$$\int_a^b f_i(t,x(t),\dot{x}(t))\mathrm{d}t \ \leq \ \int_a^b \big[f_i(t,y(t),\dot{y}(t)) + \mu_{J_0}(t)' g_{J_0}(t,y(t),\dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t,y(t),\dot{y}(t))\big]\mathrm{d}t. \tag{3.1}$$

But $\mu_{J_0}(t)' g_{J_0}(t,x(t),\dot{x}(t)) \leq 0$ and $\nu_{S_0}(t)' h_{S_0}(t,x(t),\dot{x}(t)) = 0, \quad \forall t \in I$, and from (3.1), we obtain

$$\int_a^b L_i(t,x(t),\dot{x}(t))\mathrm{d}t \leqq \int_a^b L_i(t,y(t),\dot{y}(t))\mathrm{d}t, \quad i = \overline{1,p}. \tag{3.2}$$

According to hypothesis a), relation (3.2) implies

$$b(x,y)\int_a^b \big[\eta L_{ix}(t,y(t),\dot{y}(t)) + (D\eta)' L_{i\dot{x}}(t,y(t),\dot{y}(t))\big] \leqq -\rho_i' b(x,y)\|\theta(x,y)\|^2. \tag{3.3}$$

Consider $\lambda = (\lambda_1,\ldots,\lambda_p) \geq 0$ such that $e'\lambda = 1$. Multiplying (3.3) by λ_i and summing over $i = \overline{1,p}$, we obtain

$$b(x,y)\int_a^b \big[\eta(\lambda' L_x)(t,y(t),\dot{y}(t)) + (D\eta)'(\lambda' L_{\dot{x}})(t,y(t),\dot{y}(t))\big]\mathrm{d}t \leqq -\left(\sum_{i=1}^p \lambda_i \rho_i'\right) b(x,y)\|\theta(x,y)\|^2. \tag{3.4}$$

From the constraints of $\mathcal{D}$ and Δ, we have

$$\int_a^b \big[\mu_{J_\alpha}(t)' g_{J_\alpha}(t,x(t),\dot{x}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t,x(t),\dot{x}(t))\big]\mathrm{d}t \leqq \int_a^b \big[\mu_{J_\alpha}(t)' g_{J_\alpha}(t,y(t),\dot{y}(t)) + \nu_{S_\alpha}(t)' h_{S_\alpha}(t,y(t),\dot{y}(t))\big]\mathrm{d}t$$

and according to b), we obtain

$$b(x,y)\int_a^b \big[\eta(\mu_{J_\alpha}(t)'(g_{J_\alpha})_y(t,y(t),\dot{y}(t)) + \nu_{S_\alpha}(t)'(h_{S_\alpha})_y(t,y(t),\dot{y}(t))\big]\mathrm{d}t + b(x,y)\int_a^b (D\eta)'\big[\mu_{J_\alpha}(t)'(g_{J_\alpha})_{\dot{x}}(t,y(t),\dot{y}(t)) + \nu_{S_\alpha}(t)'(h_{S_\alpha})_{\dot{x}}(t,y(t),\dot{y}(t))\big]\mathrm{d}t \leqq -\rho_\alpha'' b(x,y)\|\theta(x,y)\|. \tag{3.5}$$

Summing (3.4) and (3.5) side by side, we obtain

$$b(x,y)\int_a^b \Big[\eta\big[\lambda' f_x(t,y(t),\dot{y}(t)) + \mu(t)' g_x(t,y(t),\dot{y}(t)) + \nu(t)' h_x(t,y(t),\dot{y}(t))\big] + (D\eta)'\big[\lambda' f_{\dot{x}}(t,y(t),\dot{y}(t)) + \mu(t)' g_{\dot{x}}(t,y(t),\dot{y}(t)) + \nu(t)' h_{\dot{x}}(t,y(t),\dot{y}(t))\big]\Big]\mathrm{d}t \leqq \left(\sum_{i=1}^p \lambda_i \rho_i' + \sum_{\alpha=1}^r \rho_\alpha''\right) b(x,y)\|\theta(x,y)\|^2. \tag{3.6}$$

Taking into account the hypothesis c) at inequality (3.6) it follows $b(x, y) > 0$ and consequently (3.6) becomes

$$\int_a^b \Big[\eta\big[\lambda' f_x(t, y(t), \dot{y}(t)) + \mu(t)' g_x(t, y(t), \dot{y}(t)) + \nu(t)' h_x(t, y(t), \dot{y}(t))\big]$$
$$+(D\eta)'\big[\lambda' f_{\dot{x}}(t, y(t), \dot{y}(t)) + \mu(t)' g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)' h_{\dot{x}}(t, y(t), \dot{y}(t))\big]\Big] \mathrm{d}t$$
$$< \Big(\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha''\Big) b(x, y) \|\theta(x, y)\|^2. \tag{3.7}$$

Denoting $V = \lambda' f + \mu(t)' g + \nu(t)' h$ relation (3.7) becomes

$$\int_a^b \big[\eta' V_x + (D\eta)' V_{\dot{x}}\big] \mathrm{d}t < -\|\theta(x, y)\|^2 \Big(\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha''\Big). \tag{3.8}$$

Integrating by parts, the foregoing inequality becomes

$$\eta' V_{\dot{x}}\Big|_a^b + \int_a^b \eta' \left[V_x - \frac{\mathrm{d}}{\mathrm{d}t} V_{\dot{x}}\right] \mathrm{d}t < -\|\theta(x, y)\|^2 \Big(\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha''\Big). \tag{3.9}$$

But $\eta(a, y(a), \dot{y}(a)) = \eta(b, y(b), \dot{y}(b)) = 0$ and taking into account the first constraint of problem (MWD), relation (3.9) reduces to

$$0 < -\|\theta(x, y)\|^2 \Big(\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha''\Big).$$

According to hypothesis d), this inequality becomes $0 < 0$, that is false. □

Theorem 3.2 (Direct duality)**.** *Let x^0 be a normal efficient solution of the primal* (MP) *and suppose satisfied the hypotheses of Theorem* 3.1 *at the point x^0. Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual* (MWD)*. Moreover, we have $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Proof. Since the point x^0 is a normal efficient solution of problem (MP), according to Theorem 2.1 there are a vector $\lambda^0 \in \mathbb{R}^n$ and the piecewise non-smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ which satisfy relations (MV). The vector relation $\mu^0(t)' g(t, x^0(t), \dot{x}^0(t)) = 0$ from (MV) is equivalent to $\mu_j^0(t)' g_j(t, x^0(t), \dot{x}^0(t)) = 0$, $j = \overline{1, m}$. It follows $\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x^0(t), \dot{x}^0(t)) = 0$ for $\alpha = \overline{1, r}$. Also, we remark that $\nu_{S_\alpha}^0(t)' h_{S_\alpha}(t, x^0(t), \dot{x}^0(t)) = 0$, for $\alpha = \overline{1, r}$. Therefore, we obtain

$$\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_\alpha}^0(t)' h_{S_\alpha}(t, x^0(t), \dot{x}^0(t)) = 0 \text{ for } \alpha = \overline{1, r}$$

and then, we get $(x^0, \lambda^0, \mu^0, \nu^0) \in \Delta$. Moreover,

$$\mu_{J_0}(t)' g_{J_0}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_0}^0(t)' h_{S_0}(t, x^0(t), \dot{x}^0(t)) = 0$$

and consequently, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)\big[= f(t, x^0(t), \dot{x}^0(t))\big]$. □

Theorem 3.3 (Converse duality). *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of the dual* (MWD). *Suppose that the following conditions are satisfied:*

i) $x^0 \in \mathcal{D}$.

a) *For each* $i = \overline{1,p}$, *the functional* $\int_a^b f_i(t, x(t), \dot{x}(t))\mathrm{d}t$ *is* (ρ_i', b)*-quasiinvex at the point* x^0 *with respect to* η *and* θ.

b) *For each* $\alpha = \overline{1,r}$, $\int_a^b \left[\mu^0_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu^0_{S_\alpha}(t)' h_{S_\alpha}(t, x(t), \dot{x}(t))\right]\mathrm{d}t$ *is* (ρ_α''', b)*-quasiinvex at the point* x^0 *with respect to* η *and* θ.

c) *At least one of the integrals from* a)-b) *is strictly* (ρ, b)*-quasiinvex at the point* x^0 *with respect to* η *and* θ.

d) $\sum_{i=1}^{p} \lambda_i^0 \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha'' \geqq 0$.

Then x^0 *is an efficient solution of* (MP). *Moreover,* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Proof. On the contrary, suppose that the point x^0 is not an efficient solution of problem (MP) and we will find a contradiction. Then, there exists $x \in \mathcal{D}$ such that $(f_1, \ldots, f_p) \leq (L_1, \ldots, L_p)$. Repeating the proof of Theorem 3.1, with $(x^0, \lambda^0, \mu^0, \nu^0)$ instead of (y, λ, μ, ν), we obtain

$$0 < -\|\theta(x^0, y)\|^2 \left(\sum_{i=1}^{p} \lambda_i^0 \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha'' \right),$$

that is $0 < 0$. It means that the above made supposition is false. Therefore x^0 is an efficient solution of problem (MP), and we obtain $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$. □

• CASE STUDY 2

Theorem 3.4 (Weak duality). *Let* $x \in \mathcal{D}$ *and* $(y, \lambda, \mu, \nu) \in \Delta$ *be feasible points of problems* (MP) *and* (MWD). *Assume satisfied the following conditions:*

a) *For each* $i = \overline{1,p}$, *the functional* $\int_a^b L_i(t, x(t), \dot{x}(t))\mathrm{d}t$ *is* (ρ_i', b)*-quasiinvex at the point* y *with respect to* η *and* θ.

b) *For each* $\alpha = \overline{1,r}$, $\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t))\mathrm{d}t$ *is* (ρ_α'', b)*-quasiinvex at the point* y *with respect to* η *and* θ.

c) *For each* $\alpha = \overline{1,r}$, *the functional* $\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t))\mathrm{d}t$ *is monotonic* (ρ_α''', b)*-quasiinvex at the point* y *with respect to* η *and* θ.

d) *At least one of the integrals from* a)-c) *is strictly* (ρ, b)*-quasiinvex at the point* y *with respect to* η *and* θ.

e) $\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} (\rho_\alpha'' + \rho_\alpha''') \geqq 0$.

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ *is false.*

Theorem 3.5 (Direct duality). *Let x^0 be a normal efficient solution of the primal* (MP) *and suppose satisfied the hypotheses of Theorem* 3.4 *at the point x^0. Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual* (MWD) *and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Theorem 3.6 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of the dual* (MD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *The hypotheses* a)-e) *of Theorem* 3.4 *hold for $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$.*

Then x^0 is an efficient solution of (MP). *Moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

• CASE STUDY 3

Theorem 3.7 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems* (MP) *and* (MWD). *Assume satisfied the following conditions:*

a) *The functional $\displaystyle\int_a^b \lambda' f_x(t, x(t), \dot{x}(t))\mathrm{d}t$ is (ρ, b)-quasiinvex at the point y with respect to η and θ.*

b) *For each $\alpha = \overline{1, r}$, $\displaystyle\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t))\mathrm{d}t$ is (ρ'_α, b)-quasiinvex at the point y with respect to η and θ.*

c) *For each $\alpha = \overline{1, r}$, the functional $\displaystyle\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t))\mathrm{d}t$ is monotonic (ρ''_α, b)-quasiinvex at the point y with respect to η and θ.*

d) *At least one of the integrals from* a)-c) *is strictly (ρ, b)-quasiinvex at the point y with respect to η and θ.*

e) $\displaystyle\rho + \sum_{\alpha=1}^{r} (\rho'_\alpha + \rho''_\alpha) \geqq 0, \quad \rho \in \mathbb{R}^p.$

Then the inequality $\pi(x) \leq \delta(t, \lambda, \mu, \nu)$ is false.

Theorem 3.8 (Direct duality). *Let x^0 be a normal efficient solution of the primal* (MP) *and suppose satisfied the hypotheses of Theorem* 3.7 *at the point x^0. Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of the dual* (MWD) *and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Theorem 3.9 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution of the dual* (MWD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

ii) *The hypotheses* a)-e) *of Theorem* 3.7 *hold for $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$.*

Then x^0 is an efficient solution of (MP) *and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

• CASE STUDY 4

Theorem 3.10 (Weak duality)**.** *Let* $x \in \mathcal{D}$ *and* $(y, \lambda, \mu, \nu) \in \Delta$ *be feasible points of problems* (MP) *and* (MWD)*. Assume satisfied the following conditions:*

a) *The functional* $\int_a^b \lambda' f_x(t, x(t), \dot{x}(t))\mathrm{d}t$ *is* (ρ, b)*-quasiinvex at the point* y *with respect to* η *and* θ*.*

b) *For each* $\alpha = \overline{1, r}$, $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t))]\mathrm{d}t$ *is* (ρ'_α, b)*-quasiinvex at the point* y *with respect to* η *and* θ*.*

c) *At least one of the integrals from* a)-b) *is strictly* (ρ, b)*-quasiinvex at the point* y *with respect to* η *and* θ*.*

d) $\rho + \sum_{\alpha=1}^{r} \rho'_\alpha \geqq 0, \quad \rho \in \mathbb{R}^p.$

Then the relation $\pi(x) \leq \delta(t, \lambda, \mu, \nu)$ *is false.*

Theorem 3.11 (Direct duality)**.** *Let* x^0 *be a normal efficient solution of primal* (MP) *and suppose satisfied the hypotheses of Theorem* 3.10 *at the point* x^0*. Then there are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^q$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution of the dual* (MWD) *and moreover,* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$*.*

Theorem 3.12 (Converse duality)**.** *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of dual* (MWD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$*.*

ii) *The hypotheses* a)-d) *of Theorem* 3.10 *hold for* $(y, \lambda, \mu, \nu) = (x^0, \lambda^0, \mu^0, \nu^0)$*.*

Then the point x^0 *is an efficient solution of* (MP)*, and* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$*.*

4. FRACTIONAL DUAL MOND-WEIR-ZALMAI WITH LAGRANGIANS FOR PROBLEM (MFP)

Let $\{J_0, J_1, \ldots, J_r\}$ be a partition of $\{1, \ldots, m\}$ and $\{S_0, S_1, \ldots, S_r\}$ a partition of $\{1, \ldots, q\}$. We denote

$$F_i(y) = \int_a^b f_i(t, y(t), \dot{y}(t))\mathrm{d}t, \quad K_i(y) = \int_a^b k_i(t, y(t), \dot{y}(t))\mathrm{d}t,$$

$$\bar{F}_i(t, y(t), \dot{y}(t)) = f_i(t, y(t), \dot{y}(t)) + \mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t)).$$

Consider a function $y \in X$. We associate to problem (MFP) the following multiobjective variational problem:

$$(\text{MFZD}) \begin{cases} \text{Maximize } \left(\dfrac{\displaystyle\int_a^b \bar{F}_1(t, y(t), \dot{y}(t))\mathrm{d}t}{\displaystyle\int_a^b k_1(t, y(t), \dot{y}(t))\mathrm{d}t}, \dots, \dfrac{\displaystyle\int_a^b \bar{F}_p(t, y(t), \dot{y}(t))\mathrm{d}t}{\displaystyle\int_a^b k_p(t, y(t), \dot{y}(t))\mathrm{d}t} \right) \\ \text{subject to } y(a) = y_0, \quad y(b) = b_0, \\ \displaystyle\sum_{i=1}^{p} \lambda_i \big[K_i(y) f_{ix}(t, y(t), \dot{y}(t)) - F_i(y) k_{ix}(t, y(t), \dot{y}(t))\big] \\ \qquad + \mu(t)' g_x(t, y(t), \dot{y}(t)) + \nu(t)' h_x(t, y(t), \dot{y}(t)) = \\ = \dfrac{\mathrm{d}}{\mathrm{d}t} \Big\{ \displaystyle\sum_{i=1}^{p} \lambda_i \big[K_i(y) f_{i\dot{x}}(t, y(t), \dot{y}(t)) - F_i(y) k_{i\dot{x}}(t, y(t), \dot{y}(t))\big] \\ \qquad + \mu(t)' g_{\dot{x}}(t, y(t), \dot{y}(t)) + \nu(t)' h_{\dot{x}}(t, y(t), \dot{y}(t)) \Big\} \\ \mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{S_\alpha}(t) h_{S_\alpha}(t, y(t), \dot{y}(t)) \geqq 0, \ \alpha = \overline{1, r}, \ \forall t \in I \\ \lambda \geq 0, \quad e'\lambda = 1 \end{cases}$$

We denote by $\pi(x)$ the value of problem (MFP) at the point $x \in \mathcal{D}$ and by $\delta(y, \lambda, \mu, \nu)$ the value of dual (MFZD) at the point $(y, \lambda, \mu, \nu) \in \Delta$, where Δ is the domain of (MFZD).

• CASE STUDY 1

Theorem 4.1 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \lambda, \mu, \nu) \in \Delta$ be feasible points of problems* (MFP) *and* (MFZD). *Assume that the following conditions are satisfied:*

a) $K_i(x)\big[\mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t') h_{S_0}(t, y(t), \dot{y}(t))\big] \leqq 0$, $\forall x \in \mathcal{D}$, $\forall t \in I$, *for* $i = \overline{1, p}$.

b) *For each* $i = \overline{1, p}$, *the integral* $\displaystyle\int_a^b \big[K_i(y) f_i(t, x(t), \dot{x}(t)) - F_i(y) k_i(t, x(t), \dot{x}(t))\big]\mathrm{d}t$ *is (ρ_i', b)-quasiinvex at the point y with respect to η and θ.*

c) *For each* $\alpha = \overline{1, r}$, $\displaystyle\int_a^b \big[\mu_{J_\alpha}(t)' g_{J_\alpha}(t, y(t), \dot{y}(t)) + \nu_{K_\alpha}(t') h_{K_\alpha}(t, y(t), \dot{y}(t))\big]\mathrm{d}t$ *is (ρ_i'', b)-quasiinvex at the point y with respect to η and θ.*

d) *At least one of the integrals from* a)-b) *is strictly (ρ, b)-quasiinvex at the point y with respect to η and θ.*

e) $\displaystyle\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha'' \geqq 0$.

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$ is false.

Proof. By reductio ad absurdum, we have

$$\left(\frac{\int_a^b f_1(t,x(t),\dot{x}(t))\mathrm{d}t}{\int_a^b k_1(t,x(t),\dot{x}(t))\mathrm{d}t},\dots,\frac{\int_a^b f_p(t,x(t),\dot{x}(t))\mathrm{d}t}{\int_a^b k_p(t,x(t),\dot{x}(t))\mathrm{d}t}\right)$$
$$\leq\left(\frac{\int_a^b \bar{F}_1(t,x(t),\dot{x}(t))\mathrm{d}t}{\int_a^b k_1(t,x(t),\dot{x}(t))\mathrm{d}t},\dots,\frac{\int_a^b \bar{F}_p(t,x(t),\dot{x}(t))\mathrm{d}t}{\int_a^b k_p(t,x(t),\dot{x}(t))\mathrm{d}t}\right)$$

or componentwise,

$$\frac{\int_a^b f_i(t,x(t),\dot{x}(t))\mathrm{d}t}{\int_a^b k_i(t,x(t),\dot{x}(t))\mathrm{d}t}\leq\frac{\int_a^b \bar{F}_i(t,y(t),\dot{y}(t))\mathrm{d}t}{\int_a^b k_i(t,y(t),\dot{y}(t))\mathrm{d}t}=$$
$$=\frac{\int_a^b\big[f_i(t,y(t),\dot{y}(t))+\mu_{J_0}(t)g_{J_0}(t,y(t),\dot{y}(t))+\nu_{S_0}(t)h_{S_0}(t,y(t),\dot{y}(t))\big]\mathrm{d}t}{\int_a^b k_i(t,y(t),\dot{y}(t))\mathrm{d}t}$$

for $i=\overline{1,p}$. From this inequality it follows

$$\int_a^b K_i(y)f_i(t,x(t),\dot{x}(t))\mathrm{d}t\leq\int_a^b F_i(y)k_i(t,x(t),\dot{x}(t))\mathrm{d}t$$
$$+\int_a^b K_i(x)\big[\mu_{J_0}(t)g_{J_0}(t,y(t),\dot{y}(t))+\nu_{S_0}(t)h_{S_0}(t,y(t),\dot{y}(t))\big]\mathrm{d}t$$

and taking into account the hypothesis a) we get

$$\int_a^b K_i(y)f_i(t,x(t),\dot{x}(t))\mathrm{d}t\leq\int_a^b F_i(y)k_i(t,x(t),\dot{x}(t))\mathrm{d}t.$$

We have

$$\int_a^b\big[K_i(y)f_i(t,x(t),\dot{x}(t))-F_i(y)k_i(t,x(t),\dot{x}(t))\big]\mathrm{d}t\leqq 0$$
$$=\int_a^b\big[K_i(y)f_i(t,y(t),\dot{y}(t))-F_i(y)k_i(t,y(t),\dot{y}(t))\big]\mathrm{d}t \tag{4.1}$$

or $F_i(x)K_i(y)-K_i(x)F_i(y)\leqq 0$.

According to b), relation (4.1) implies

$$b(x,y)\int_a^b \Big\{\eta'\big[K_i(y)f_{ix}(t,y(t),\dot{y}(t))-F_i(y)k_{ix}(t,y(t),\dot{y}(t))\big]$$
$$+(D\eta)'\big[K_i(y)f_{i\dot{x}}(t,y(t),\dot{y}(t))-F_i(y)k_{i\dot{x}}(t,y(t),\dot{y}(t))\big]\Big\}\mathrm{d}t$$
$$\leqq -\rho_i' b(x,y)\|\theta(x,y)\|. \tag{4.2}$$

Multiplying (4.1) and (4.2) by λ_i, $i=\overline{1,p}$ $(\lambda \geq 0)$ and summing over $i=\overline{1,p}$, we obtain

$$\sum_{i=1}^{p}\lambda_i\big[F_i(x)K_i(y)-K_i(x)F_i(y)\big]\leqq 0 \Rightarrow$$
$$b(x,y)\int_a^b \Big\{\eta'\sum_{i=1}^{p}\lambda_i\big[K_i(y)f_{ix}(t,y(t),\dot{y}(t))-F_i(y)k_{ix}(t,y(t),\dot{y}(t))\big]$$
$$+(D\eta)'\sum_{i=1}^{p}\lambda_i\big[K_i(y)f_{i\dot{x}}(t,y(t),\dot{y}(t))-F_i(y)k_{i\dot{x}}(t,y(t),\dot{y}(t))\big]\Big\}\mathrm{d}t$$
$$\leqq -b(x,y)\|\theta(x,y)\|^2\sum_{i=1}^{p}\lambda_i\rho_i'. \tag{4.3}$$

According to c), we have

$$\int_a^b \big[\mu_{J_\alpha}(t)'g_{J_\alpha}(t,x(t),\dot{x}(t))+\nu_{S_\alpha}(t)'h_{S_\alpha}(t,x(t),\dot{x}(t))\big]\mathrm{d}t$$
$$\leqq \int_a^b \big[\mu_{J_\alpha}(t)'g_{J_\alpha}(t,y(t),\dot{y}(t))+\nu_{S_\alpha}(t)'h_{S_\alpha}(t,y(t),\dot{y}(t))\big]\mathrm{d}t$$
$$\Rightarrow b(x,y)\int_a^b \Big\{\eta'\big[\mu_{J_\alpha}(t)'\left(g_{J_\alpha}\right)_x(t,y(t),\dot{y}(t))+\nu_{S_\alpha}(t)'\left(h_{S_\alpha}\right)_x(t,y(t),\dot{y}(t))\big]$$
$$+(D\eta)'\big[\mu_{J_\alpha}(t)'\left(g_{J_\alpha}\right)_{\dot{x}}(t,y(t),\dot{y}(t))+\nu_{S_\alpha}(t)'\left(h_{S_\alpha}\right)_{\dot{x}}(t,y(t),\dot{y}(t))\big]\Big\}\mathrm{d}t$$
$$\leqq -\rho_\alpha'' b(x,y)\|\theta(x,y)\|^2. \tag{4.4}$$

Summing side by side over $\alpha=\overline{1,r}$ the double implication (4.4), we obtain

$$\int_a^b \big[\mu(t)'g(t,x(t),\dot{x}(t))+\nu(t)'h(t,x(t),\dot{x}(t))\big]\mathrm{d}t$$
$$-\int_a^b \big[\mu(t)'g(t,y(t),\dot{y}(t))+\nu(t)'h(t,y(t),\dot{y}(t))\big]\mathrm{d}t\leqq 0 \quad \Rightarrow$$

$$\Rightarrow \quad b(x,y)\int_a^b \Big\{\eta'\big[\mu(t)'g_x(t,y(t),\dot y(t))+\nu(t)'h_x(t,y(t),\dot y(t))\big]$$
$$+(D\eta)'\big[\mu(t)'g_{\dot x}(t,y(t),\dot y(t))+\nu(t)'h_{\dot x}(t,y(t),\dot y(t))\big]\Big\}\mathrm{d}t$$
$$\leqq -b(x,y)\|\theta(x,y)\|^2\sum_{\alpha=1}^{r}\rho''_\alpha. \tag{4.5}$$

Summing now side by side the double implications (4.3) and (4.5) and taking into account c) it follows

$$\sum_{i=1}^{p}\lambda_i\big[F_i(x)K_i(y)-K_i(x)F_i(y)\big]+\int_a^b\big[\mu(t)'g(t,x(t),\dot x(t))+\nu(t)'h(t,x(t),\dot x(t))\big]\mathrm{d}t$$
$$-\int_a^b\big[\mu(t)'g(t,y(t),\dot y(t))+\nu(t)'h(t,y(t),\dot y(t))\big]\mathrm{d}t\leqq 0$$
$$\Rightarrow b(x,y)\int_a^b\Big\{\eta'\Big[\sum_{i=1}^{p}\lambda_i\big[K_i(y)f_{ix}(t,y(t),\dot y(t))-F_i(y)k_{ix}(t,y(t),\dot y(t))\big]$$
$$+\mu(t)'g_x(t,y(t),\dot y(t))+\nu(t)'h_x(t,y(t),\dot y(t))\big]\Big]$$
$$+(D\eta)'\Big[\sum_{i=1}^{p}\lambda_i\big[K_i(y)f_{i\dot x}(t,y(t),\dot y(t))-F_i(y)k_{i\dot x}(t,y(t),\dot y(t))\big]$$
$$+\mu(t)'g_{\dot x}(t,y(t),\dot y(t))+\nu(t)'h_{\dot x}(t,y(t),\dot y(t))\big]\Big]\Big\}dt$$
$$< -b(x,y)\|\theta(x,y)\|^2\Big(\sum_{i=1}^{p}\lambda_i\rho'_i+\sum_{\alpha=1}^{r}\rho''_\alpha\Big). \tag{4.6}$$

By the second implication of (4.6) we get $b(x,y)>0$. Then, the second implication of (4.6), shortly, becomes

$$\int_a^b\big[\eta'V_x+(D\eta)'V_{\dot x}\big]\mathrm{d}t<-\|\theta(x,y)\|^2\Big(\sum_{i=1}^{p}\lambda_i\rho'_i+\sum_{\alpha=1}^{r}\rho''_\alpha\Big), \tag{4.7}$$

where we denoted

$$V \;=\; \sum_{i=1}^{p}\lambda_i\big[K_i(y)f_i(t,y(t),\dot y(t))-F_i(y)k_i(t,y(t),\dot y(t))\big]$$
$$+\mu(t)'g(t,y(t),\dot y(t))+\nu(t)'h(t,y(t),\dot y(t)).$$

Using an integration by parts, the previous inequality becomes

$$\eta'V_{\dot x}\Big|_a^b+\int_a^b\eta'\Big[V_x-\frac{\mathrm{d}}{\mathrm{d}t}V_{\dot x}\Big]\mathrm{d}t<-\|\theta(x,y)\|^2\Big(\sum_{i=1}^{p}\lambda_i\rho'_i+\sum_{\alpha=1}^{r}\rho''_\alpha\Big) \tag{4.8}$$

But $\eta(a, y(a), \dot{y}(a)) = \eta(b, y(b), \dot{y}(b)) = 0$ and taking into account the first constraint of problem (MFZD), relation (4.8) reduces to

$$0 < -\|\theta(x, y)\|^2 \left(\sum_{i=1}^{p} \lambda_i \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha'' \right).$$

Taking into account now the hypothesis d), this inequality becomes $0 < 0$, that is false. Consequently, the above made supposition is false. □

Theorem 4.2 (Direct duality). *Let x^0 be a normal efficient solution of primal* (MFP) *and suppose satisfied the hypotheses of Theorem* 4.1 *at x^0. Then there are $\lambda^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^r$ such that $(x^0, \lambda^0, \mu^0, \nu^0)$ is an efficient solution of dual* (MFZD) *and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Proof. Because x^0 is a normal efficient solution of problem (MFP), according to Theorem 2.2 there are a vector $\lambda^0 \in \mathbb{R}^n$ and the piecewise non-smooth functions $\mu^0 \colon I \to \mathbb{R}^m$ and $\nu^0 \colon I \to \mathbb{R}^q$ which satisfy relations (MFV).

Vector relation $\mu^0(t)' g(t, x^0(t), \dot{x}^0(t)) = 0$ is equivalent to relations

$$\mu_j^0(t)' g_j(t, x^0(t), \dot{x}^0(t)) = 0, \quad j = \overline{1, m},$$

from which it results $\mu_{J_\alpha}^0(t)' g_{J_\alpha}(t, x^0(t), \dot{x}^0(t)) = 0, \quad \alpha = \overline{0, r}$.

Also $\nu_{S_\alpha}^0(t)' h_{S_\alpha}(t, x^0, \dot{x}^0) = 0$, $\alpha = \overline{0, r}$. Then we have

$$\mu_{J_\alpha}^0(t)' g_{J_\alpha}(t, x^0(t), \dot{x}^0(t)) + \nu_{S_\alpha}^0(t)' h_{S_\alpha}(t, x^0(t), \dot{x}^0(t)) = 0, \quad \alpha = \overline{0, r},$$

that is $(x^0, \lambda^0, \mu^0, \nu^0) \in \Delta$ and in addition, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$. □

Theorem 4.3 (Converse duality). *Let $(x^0, \lambda^0, \mu^0, \nu^0)$ be an efficient solution to dual* (MFZD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

a) $K_i(x^0)\left[\mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t))\right] \leqq 0$, $\forall x \in \mathcal{D}$, $\forall t \in I$, $i = \overline{1, p}$.

b) *For each* $i = \overline{1, p}$, $\displaystyle\int_a^b \left[K_i(x^0) f_i(t, x(t), \dot{x}(t)) - F_i(x^0) k_i(t, x(t), \dot{x}(t))\right] \mathrm{d}t$ *is (ρ_i', b)-quasiinvex at the point $x = x^0$ with respect to η and θ.*

c) *For each* $\alpha = \overline{1, r}$, $\displaystyle\int_a^b \left[\mu_{J_\alpha}^0(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) + \nu_{S_\alpha}^0(t)' h_{S_\alpha}(t, x(t), \dot{x}(t))\right] \mathrm{d}t$ *is (ρ_α'', b)-quasiinvex at the point $x = x^0$ with respect to η and θ.*

d) *At least one of the integrals from* a)-c) *is strictly (ρ, b)-quasiinvex at the point $x = x^0$ with respect to η and θ.*

e) $\displaystyle\sum_{i=1}^{p} \lambda_i^0 \rho_i' + \sum_{\alpha=1}^{r} \rho_\alpha'' \geqq 0.$

Then x^0 is an efficient solution to (MFP) *and moreover, $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.*

Proof. On the contrary, suppose that the point x^0 is an efficient solution of problem (MFP) and then, we will find a contradiction. Then there exists $x \in \mathcal{D}$ such that for each $i = \overline{1,p}$:

$$\frac{\displaystyle\int_a^b f_i(t,x(t),\dot{x}(t))\mathrm{d}t}{\displaystyle\int_a^b k_i(t,x(t),\dot{x}(t))\mathrm{d}t} \leq$$

$$\leq \frac{\displaystyle\int_a^b f_i(t,x^0(t),\dot{x}^0(t)) + \mu_{J_0}(t)' g_{J_0}(t,x^0(t),\dot{x}^0(t)) + \nu_{S_0}(t)' h_{S_0}(t,x^0(t),\dot{x}^0(t))\big]\mathrm{d}t}{\displaystyle\int_a^b k_i(t,x^0(t),\dot{x}^0(t))\mathrm{d}t}.$$

Following the proof of Theorem 4.1, with $(x^0,\lambda^0,\mu^0,\nu^0)$ instead of (y,λ,μ,ν), we obtain

$$0 < -\|\theta(x,x^0)\|^2\Big(\sum_{i=1}^p \lambda_i^0\rho_i' + \sum_{\alpha=1}^r \rho_\alpha''\Big),$$

that yields $0 < 0$. It means that the above made supposition is false. Therefore the point x^0 is an efficient solution of problem (MFP) and we have $\pi(\bar{x}) = \delta(x^0,\lambda^0,\mu^0,\nu^0)$. □

• CASE STUDY 2

Theorem 4.4 (Weak duality)**.** *Let* $x \in \mathcal{D}$ *and* $(y,\lambda,\mu,\nu) \in \Delta$ *be feasible points of problems* (MFP) *and* (MFZD)*. Assume satisfied the following conditions:*

a) $K_i(x)\big[\mu_{J_0}(t)' g_{J_0}(t,y(t),\dot{y}(t)) + \nu_{S_0}(t')h_{S_0}(t,y(t),\dot{y}(t))\big] \leqq 0$, $\forall x \in \mathcal{D}$, $\forall t \in I$, $i = \overline{1,p}$.

b) *For each* $i = \overline{1,p}$*, the integral* $\displaystyle\int_a^b \big[K_i(y)f_i(t,x(t),\dot{x}(t)) - F_i(y)k_i(t,x(t),\dot{x}(t))\big]\mathrm{d}t$ *is* (ρ_i',b)*-quasiinvex at* y *with respect to* η *and* θ*.*

c) *For each* $\alpha = \overline{1,r}$*, the functional* $\displaystyle\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}(t,x(t),\dot{x}(t))\mathrm{d}t$ *is* (ρ_i'',b)*-quasiinvex at* y *with respect to* η *and* θ*.*

d) *For each* $\alpha = \overline{1,r}$*,* $\displaystyle\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}(t,x(t),\dot{x}(t))\mathrm{d}t$ *is monotonic* (ρ_i''',b)*-quasiinvex at the point* y *with respect to* η *and* θ*.*

e) *At least one of the integrals from* a)-b) *is strictly* (ρ,b)*-quasiinvex at* y *with respect to* η *and* θ*.*

f) $\displaystyle\sum_{i=1}^p \lambda_i\rho_i' + \sum_{\alpha=1}^r(\rho_\alpha'' + \rho_\alpha''') \geqq 0.$

Then the inequality $\pi(x) \leq \delta(y,\lambda,\mu,\nu)$ *is false.*

Theorem 4.5 (Direct duality)**.** *Let* x^0 *be a normal efficient solution of primal* (MFP) *and suppose satisfied the hypotheses of Theorem* 4.4 *at* x^0*. Then there* *are* $\lambda^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu^0 \colon I \to \mathbb{R}^m$ *and* $\nu^0 \colon I \to \mathbb{R}^r$ *such that* $(x^0, \lambda^0, \mu^0, \nu^0)$ *is an efficient solution of dual* (MFZD) *and moreover,* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Theorem 4.6 (Converse duality). *Let* $(x^0, \lambda^0, \mu^0, \nu^0)$ *be an efficient solution of dual* (MFZD) *and suppose satisfied the following conditions:*

i) $x^0 \in \mathcal{D}$.

a) $K_i(x^0)\left[\mu_{J_0}(t)' g_{J_0}(t, y(t), \dot{y}(t)) + \nu_{S_0}(t)' h_{S_0}(t, y(t), \dot{y}(t))\right] \leqq 0$, $\forall t \in I$, $i = \overline{1,p}$.

b) *For each* $i = \overline{1,p}$, $\int_a^b \left[K_i(x^0) f_i(t, x(t), \dot{x}(t)) - F_i(x^0) k_i(t, x(t), \dot{x}(t))\right] \mathrm{d}t$ *is* (ρ'_i, b)*-quasiinvex at the point* $x = x^0$ *with respect to* η *and* θ.

c) *For each* $\alpha = \overline{1,r}$, *the integral* $\int_a^b \mu_{J_\alpha}(t)' g_{J_\alpha}(t, x(t), \dot{x}(t)) \mathrm{d}t$ *is* (ρ''_α, b)*-quasiinvex at the point* $x = x_0$ *with respect to* η *and* θ.

d) *For each* $\alpha = \overline{1,r}$, *the integral* $\int_a^b \nu_{K_\alpha}(t)' h_{K_\alpha}(t, x(t), \dot{x}(t)) \mathrm{d}t$ *is monotonic* (ρ'''_α, b)*-quasiinvex at* $x = x_0$ *with respect to* η *and* θ.

e) *One of the integrals from* a)-c) *is strictly* (ρ, b)*-quasiinvex at* x^0 *with respect to* η *and* θ.

f) $\sum_{i=1}^{p} \lambda_i^0 \rho'_i + \sum_{\alpha=1}^{r} (\rho''_\alpha + \rho'''_\alpha) \geqq 0$.

Then x^0 *is an efficient solution of* (MFP). *Moreover,* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

REFERENCES

[1] R. Jagannathan: *Duality for nonlinear fractional programming*, Z. Oper. Res., Series A, **17**(1973), 1-3.

[2] D. S. Kim and A. L. Kim: *Optimality and duality for nondifferentiable multiobjective variational problems*, J. Math. Appl., **274**(2002), 255-278.

[3] Şt. Mititelu: *Kuhn-Tucker conditions and duality for multiobjective fractional programs with n-set functions*, Proc. 7-th Balkan Conf. on Operational Research, 123-132, Constanţa, May 25-28, 2005, Bucharest, 2007.

[4] Şt. Mititelu: *Invex functions*, Rev. Roumaine Math. Pures Appl., **49**(2004), No. 5-6, 529-544.

[5] Şt. Mititelu: *Efficiency and duality for multiobjective fractional problems in optimal control with* ρ*-quasiinvexity*, Int. Conf. "Trends and Challenges in Appl. Math."- ICTCAM 2007, June 20-23, 2007, Bucharest.

[6] Şt. Mititelu: *Efficiences conditions for multiobjective fractional variational problems*, Appl. Sci., **10**(2008), 162-175.

[7] Şt. Mititelu: *Extensions in invex theory*, J. Adv. Math. Studies, **1**(2008), No. 1-2, 63-70.

[8] Şt. Mititelu: *Optimality and duality for invex multi-time control problems with mixed constraints*, J. Adv. Math. Studies, **2**(2009), No. 1, 25-34.

[9] Şt. Mititelu and I.M. Stancu-Minasian: *Efficiency and duality for multiobjective fractional variational problems with* (ρ, b)*-quasiinvexity*, Yugoslav J. Oper. Res., **19**(2009), No. 1, 85-99.

[10] B. Mond and M. A. Hanson: *Duality for variational problems*, J. Math. Anal. Appl., **18**(1967), 355-364.

[11] B. Mond, S. Chandra and I. Husain: *Duality for variational problems with invexity*, J. Math. Anal. Appl., **134**(1988), No. 2, 322-328.

[12] B. Mond and I. Husain: *Sufficient optimality criteria and duality for variational problems with generalized invexity*, J. Aust. Math. Soc., Series B, **31**(1989), No. 1, 108-121.

[13] R. N. Mukherjee and Ch. Purnachandra Rao: *Mixed type duality for multiobjective variational problems*, J. Math. Anal. Appl., **225**(2000), 571-586.

[14] Ariana Pitea, C. Udrişte and Şt. Mititelu: *PDE-constrained optimization problems with curvilinear functional quotients as objective vectors*, Balkan J. Geom. Appl., **14**(2009), No. 2, 75-88.

[15] Ariana Pitea, C. Udrişte and Şt. Mititelu: *New type dualities in PDI and PDE constrained optimization problems* , J. Adv. Math. Studies, **2**(2009), No. 1, 81-90.

[16] V. Preda: *On Mond-Weir duality for variational problems*, Rev. Roumanie Math. Pures Appl., **28**(1993), No. 2, 155-164.

[17] V. Preda: *Optimality and duality in fractional multiple objective programming semilocally preinvex and related functions*, J. Math. Anal. Appl., **288**(2003), No. 2, 365-382.

[18] I. M. Stancu-Minasian and Şt. Mititelu: *Multiobjective fractional variational problems with* (ρ, b)*-quasiinvexity*, Proc. of the Romanian Academy, Series A, **9**(2008), No. 1, 5-11.

[19] F. A. Valentine: *The problem of Lagrange with differentiable inequality as added side conditions*, 407-448, in “Contributions to the Calculus of Variations”, 1933-37, Univ. of Chicago Press, 1937.

[20] G. J. Zalmai: *Generalized* $(\mathcal{F}, b, \phi, \rho, \theta)$*-univex* n*-set functions and semiparametric duality models in multiobjective fractional subset programming*, Int. J. Math. Math. Sci., **2005**(2005), No. 7, 1109-1133.

Technical University of Civil Engineering
Bd. Lacul Tei 124, 020396 *Bucharest, Romania*
E-mail address: st_mititelu@yahoo.com

University “Politehnica” of Bucharest
Faculty of Applied Sciences
Splaiul Independenţei, No. 313, 060042 *Bucharest, Romania*
E-mail address: mihai@mathem.pub.ro