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ON SOME NEW CLASSES OF SETS AND A NEW DECOMPOSITION OF CONTINUITY VIA GRILLS

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

Abstract. In this paper, we present and study some new classes of sets and give a new decomposition of continuity in terms of grills.

1. INTRODUCTION AND PRELIMINARIES

The idea of grill on a topological space was first introduced by Choquet [4]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds ([2], [3], [9]). In [8], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. We are utilizing the same procedure in this paper. (For other advances on topological spaces obtained by our research group we recommend [1], [5]).

Throughout this paper, X or (X, τ) represent a topological space with no separation axioms assumed unless explicitly stated. For a subset A of a space X, the closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. The power set of X will be denoted by $\wp(X)$. A collection G of a nonempty subsets of a space X is called a grill [4] on X if (i) $A \in G$ and $A \subset B \Rightarrow B \in G$, (ii) $A, B \subset X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$. For any point x of a topological space (X, τ), $\tau(x)$ denote the collection of all open neighborhoods of x. Let (X, τ) be a topological space. A subset A in X is said to be a t-set ([7] and [10]) if $\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(A)$. A subset A in X is said to be a B-set [10] if there is a $U \in \tau$ and a t-set A in (X, τ)

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such that $H = U \cap A$, respectively. A subset A in X is said to be preopen [6] (resp. regular open) if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$(resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Definition 1.1 ([8])**.** Let (X, τ) be a topological space and G be a grill on X. The mapping $\Phi\colon \wp(X) \to \wp(X)$, denoted by $\Phi_G(A, \tau)$ for $A \in \wp(X)$ or simply $\Phi(A)$ called the operator associated with the grill G and the topology τ and is defined by $\Phi G(A) = \{x \in X \mid A \cap U \in G, \forall U \in \tau(x)\}$.

Proposition 1.1 ([8])**.** *Let (X, τ) be a topological space and G be a grill on X. Then for all $A, B \subset X$:*

i) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$*;*

ii) $\Phi(\Phi(A)) \subset \Phi(A) = \mathrm{Cl}(\Phi(A)) \subset \mathrm{Cl}(A)$*.*

Let G be a grill on a space X. Then a map $\Psi\colon \wp(X) \to \wp(X)$ is defined by $\Psi(A) = A \cup \Phi(A)$, for all $A \in \wp(X)$. The map Ψ satisfies Kuratowski closure axioms. Corresponding to a grill G on a topological space (X, τ), there exists a unique topology τ_G on X given by $\tau_G = \{U \subset X \mid \Psi(X - U) = X - U\}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_G - \mathrm{Cl}(A)$. For any grill G on a topological space (X, τ), $\tau \subset \tau_G$ [8]. If (X, τ) is a topological space and G is a grill on X, then we denote a grill topological space by (X, τ, G).

Let (X, τ) be a topological space and G be any grill on X. Then $A \subset B \subset X$ implies $\Phi(A) \subset \Phi(B)$ [8].

Theorem 1.1 ([8])**.** i) *If G_1 and G_2 are two grills on a space X with $G_1 \subset G_2$, then $\tau_{G1} \subset \tau_{G2}$.*

ii) *If G is a grill on a space X and $B \notin G$, then B is closed in (X, τ, G).*

iii) *For any subset A of a space X and any grill G on X, $\Phi(A)$ is τ_G-closed.*

Theorem 1.2 ([8])**.** *Let (X, τ) be a topological space and G be a grill on X. If $U \in \tau$, then $U \cap \Phi(A) = U \cap \Phi(U \cap A)$ for any $A \subset X$.*

2. SOME NEW CLASSES OF SETS

Definition 2.1. Let (X, τ) be a topological space and G be a grill on X. A subset A in X is said to be:

i) Φ-open if $A \subset \mathrm{Int}(\Phi(A))$;

ii) g-set if $\mathrm{Int}(\Psi(A)) = \mathrm{Int}(A)$;

iii) $g\Phi$-set if $\mathrm{Int}(\Phi(A)) = \mathrm{Int}(A)$.

Remark 2.1. It should be noted that:

i) Open set and Φ-open set are independent from each other.

ii) Every $g\Phi$-set is a g-set, but it is not conversely.

Example 2.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$. If $G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, then G is a grill on X such that $\tau - \{\emptyset\} \subset G$. Take $A = \{a, b, d\} \in \tau$, but it is not Φ-open, since $\Phi(\{a, b, d\}) = \{a\}$. And take $B = \{a, b\} \notin \tau$, but it is a Φ-open since $\Phi(\{a, b\}) = X$. Furthermore, $A = \{a, b, d\}$ is a g-set, but it is not a $g\Phi$-set.

Proposition 2.1. *A τ_G-closed set is equivalent to a g-set.*

Proof. Let A be a subset in (X, τ, G). Then $\Phi(A)$ is τG-closed by Theorem 1.1 (iii). $\mathrm{Int}(\Psi(\Phi(A))) = \mathrm{Int}(\Phi(A) \cup \Phi(\Phi(A))) = \mathrm{Int}(\Phi(A))$, i.e. $\Phi(A)$ is a g-set. □

Definition 2.2. A subset A of (X, τ, G) is said to be G-regular if $\mathrm{Int}(\Psi(A)) = A$

Proposition 2.2. *Every G-regular open set is a g-set.*

Proof. Obvious. □

Example 2.2 ([8])**.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. If $G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, then G is a grill on X such that $\tau - \{\emptyset\} \subset G$. Take $A = \{a, c\}$, then A is a g-set but it is not a G-regular set.

Proposition 2.3. *A t-set is a g-set.*

Proof. Let A be a t-set. Then

$$\mathrm{Int}(A) \subset \mathrm{Int}(\Psi(A)) = \mathrm{Int}(A \cup \Phi(A)) \subset \mathrm{Int}(A \cup \mathrm{Cl}(A)) = \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(A).$$

Therefore, A is a g-set. □

Remark 2.2. The converse of Proposition 2.3 is false. By the same conditions as in Example 2.2, take $A = \{a, c\}$. Then A is a g-set and also a $g\Phi$-set, but it is not a t-set.

Proposition 2.4. *If A, B are two g-sets, then $A \cap B$ is a g-set.*

Proof. $\mathrm{Int}(A \cap B) \subset \mathrm{Int}(\Psi(A \cap B)) = Int(\Psi(A \cap B) \cap \Psi(A \cap B)) = \mathrm{Int}(\Psi(A \cap B)) \cap \mathrm{Int}(\Psi(A \cap B)) \subset \mathrm{Int}(\Psi(A)) \cap \mathrm{Int}(\Psi(B)) = \mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$. Then $A \cap B$ is a g-set. □

Definition 2.3. Let (X, τ) be a topological space and G be a grill on X. A subset A in X is said to be G-preopen set if $A \subset \mathrm{Int}(\Psi(A))$.

Example 2.3. In Example 2.2, take $A = \{a, c\}$. Then A is preopen, but it is not G-preopen.

Proposition 2.5. *A G-preopen set A is a preopen set.*

Proof. Let A be a G-preopen. Then

$$A \subset \mathrm{Int}(\Psi(A)) = \mathrm{Int}(A \cup \Phi(A)) \subset \mathrm{Int}(A \cup \mathrm{Cl}(A)) = \mathrm{Int}(\mathrm{Cl}(A)).$$

Therefore, A is a preopen set. □

Remark 2.3. By Example 2.9 in [8], since if $G = \wp(X) - \{\emptyset\}$ in (X, τ), then $\tau_G = \tau$, G-preopen and preopen sets are equivalent.

Proposition 2.6. *If A is a G-preopen, then* $\mathrm{Cl}(\mathrm{Int}(\Psi(A))) = \mathrm{Cl}(A)$

Proof. $\mathrm{Cl}(A) \subset \mathrm{Cl}(\mathrm{Int}(\Psi(A))) \subset \mathrm{Cl}(\Psi(A)) = \mathrm{Cl}(A \cup \Phi(A)) = \mathrm{Cl}(A) \cup \mathrm{Cl}(\Phi(A)) = \mathrm{Cl}(A) \cup \Phi(A) \subset \mathrm{Cl}(A)$. □

Proposition 2.7. *Every Φ-open set A is G-preopen.*

Proof. Let A be a Φ-open. Then $A \subset \mathrm{Int}(\Phi(A)) \subset \mathrm{Int}(A \cup \Phi(A)) = \mathrm{Int}(\Psi(A))$. Therefore A is G-preopen. □

Proposition 2.8. *Let (X, τ, G) be a grill topological space with I arbitrary index set. Then:*

i) *If $\{A_i \mid i \in I\}$ are G-preopen sets, then $\cup\{A_i \mid i \in I\}$ is a G-preopen set.*

ii) *If A is a G-preopen set and $U \in \tau$, then $(A \cap U)$ is a G-preopen set.*

Proof. i) Let $\{A_i \mid i \in I\}$ be G-preopen sets, then $A_i \subset \mathrm{Int}(\Psi(A_i))$ for every $i \in I$. Thus

$$\cup A_i \subset \cup(\mathrm{Int}(\Psi(A_i))) \subset \mathrm{Int}(\cup(\Psi(A_i))) = \mathrm{Int}(\cup(A_i \cup \Phi(A_i))) = \mathrm{Int}(\cup A_i) \cup (\cup\Phi(A_i)))) = \mathrm{Int}(\cup A_i \cup \Phi(\cup A_i)) = \mathrm{Int}(\Psi(\cup A_i)).$$

ii) Let A be a G-preopen set and $U \in \tau$. By Theorem 1.2,

$$U \cap A \subset U \cap \mathrm{Int}(\Psi(A)) = U \cap \mathrm{Int}(A \cup \Phi(A)) = \mathrm{Int}(U \cap (A \cup \Phi(A))) = \mathrm{Int}(U \cap A) \cup (U \cap \Phi(A))) = \mathrm{Int}(U \cap A) \cup (U \cap \Phi(U \cap A))) \subset \mathrm{Int}((U \cap A) \cup \Phi(U \cap A)) = \mathrm{Int}(\Psi(U \cap A)),$$

which completes the proof. □

Definition 2.4. Let (X, τ) be a topological space and G a grill on X. A subset A in X is said to be G-set (resp. $G\Phi$-set) if there is a $U \in \tau$ and a g-set (resp. $g\Phi$-set) A in (X, τ, G) such that $H = U \cap A$, respectively.

Proposition 2.9. i) *A g-set A is a G-set.*

ii) *A $g\Phi$-set A is a $G\Phi$-set.*

Proof. Obvious. □

Proposition 2.10. *An open set U is a G-set (resp. $G\Phi$-set).*

Proof. $U = U \cap X$, $\mathrm{Int}(\Psi(X)) = \mathrm{Int}(X)$. □

Proposition 2.11. *A τ_G-closed set C is a G-set*

Proof. It follows from Proposition 2.1 and Proposition 2.9. □

Proposition 2.12. i) *A B-set is a G-set.*
ii) *A G-set is a $G\Phi$-set.*

Proof. i) Let H be a B-set. Then $H = U \cap A$, where $U \in \tau$ and A is a t-set. $H = U \cap \mathrm{Int}(A) = U \cap \mathrm{Int}(\mathrm{Cl}(A)) = U \cap \mathrm{Int}(A \cup \mathrm{Cl}(A)) \supset U \cap \mathrm{Int}(A \cup \Phi(A)) = U \cap \mathrm{Int}(\Psi(A)) \supset U \cap \mathrm{Int}(A) = H$. Therefore H is a G-set.
ii) Similar to i). □

The converse of Proposition 2.12 is false as it is shown by the following example.

Example 2.4. In Example 2.2 $A = \{a, c\}$ is a G-set and also a $G\Phi$-set, but it is not B-set. In Example 2.1, $A = \{a, b, d\}$ is a G-set, but it is not $G\Phi$-set.

Proposition 2.13. *A subset S in a space (X, τ, G) is open if and only if it is a G-preopen and a G-set.*

Proof. Necessity. It follows from Proposition 2.10 and the obvious fact that every open set is G-preopen.

Sufficiency. Since S is a G-set, then $S = U \cap A$ where U is an open set and $\mathrm{Int}(\Psi(A)) = \mathrm{Int}(A)$. Since S is also G-preopen, we have

$$\begin{aligned} S \subset \mathrm{Int}(\Psi(S)) &= \mathrm{Int}(\Psi(U \cap A)) = \mathrm{Int}(\Psi(U \cap A) \cap \Psi(U \cap A)) \subset \mathrm{Int}(\Psi(U) \cap \Psi(A)) \\ &= \mathrm{Int}(\Psi(U) \cap \mathrm{Int}(\Psi(A)) = \mathrm{Int}(U \cup \Phi(U)) \cap \mathrm{Int}(\Psi(A)) \\ &\subset \mathrm{Int}(\mathrm{Cl}(U)) \cap \mathrm{Int}(\Psi(A)) = \mathrm{Int}(\mathrm{Cl}(U)) \cap \mathrm{Int}(A). \end{aligned}$$

Hence

$$\begin{aligned} S = U \cap A &= (U \cap A) \cap U \subset (\mathrm{Int}(\mathrm{Cl}(U) \cap \mathrm{Int}(A)) \cap U \\ &= (\mathrm{Int}(\mathrm{Cl}(U)) \cap U) \cap \mathrm{Int}(A) = U \cap \mathrm{Int}(A). \end{aligned}$$

Therefore, $S = U \cap A \supset U \cap \mathrm{Int}(A)$ and $S = U \cap \mathrm{Int}(A)$. Thus S is an open set. □

Corollary 2.1. *If S is both $G\Phi$-set and Φ-open set in (X, τ, G), then S is open.*

Definition 2.5. Let (X, τ, G) be a grill space and $A \subset X$. A set A is said to be G-dense in X, if $\Psi(A) = X$.

Proposition 2.14. *A subset A of a grill G in a space (X, τ, G) is G-dense if and only if for every open set U containing $x \in X$, $A \cap U \in G$.*

Proof. Necessity. Let A be a G-dense set. Then, for every open set U containing x in a space X, $x \in \Psi(A) = A \cup \Phi(A)$. Hence if $x \in A$, then $A \cap U \in G$ and if $x \in \Phi(A)$, then $A \cap U \in G$.

Sufficiency. Let every $x \in X$. Moreover, let every open subset U of X containing x such that $A \cap U \in G$. Then if $x \in A$ or $x \in \Phi(A)$, we have $A \cap U \in G$. It follows that $x \in \Psi(A)$ and thus $X \subset \Psi(A)$. Therefore $\Psi(A) = X$. □

Proposition 2.15. *If U is an open set and A is a G-dense set in (X, τ, G), then $\Psi(U) = \Psi(U \cap A)$.*

Proof. Since $A \cap U \subset U$, we have $\Psi(U \cap A) \subset \Psi(U)$. Conversely, if $x \in \Psi(U)$, $x \in U$ and $x \in \Phi(U)$. Then for every open set V containing x, $U \cap V \in G$. Put $W = U \cap V \in \tau(x)$. Since $\Psi(A) = X$, $W \cap A \in G$, i.e. $W = (U \cap A) \cap V \in G$. Therefore, $x \in \Psi(U \cap A)$ and $\Psi(U) = \Psi(U \cap A)$. □

Proposition 2.16. *For any subset A of a space (X, τ, G), the following are equivalent:*

1. *A is G-preopen;*
2. *there is a G-regular open set U of X such that $A \subset U$ and $\Psi(A) = \Psi(U)$;*
3. *A is the intersection of G-regular open set and a G-dense set;*
4. *A is the intersection of an open set and a G-dense set.*

Proof. (1) $\Rightarrow$ (2): Let A be G-preopen in (X, τ, G), i.e. $A \subset \mathrm{Int}(\Psi(A))$. Let $U = \mathrm{Int}(\Psi(A))$. Then U is G-regular open such that $A \subset U$ and $\Psi(A) \subset \Psi(G) = \Psi(\mathrm{Int}(\Psi(A)) \subset \Psi(\Psi(A)) = \Psi(A)$. Hence $\Psi(A) = \Psi(U)$.

(2) $\Rightarrow$ (3): Suppose (2) holds. Let $D = A \cup (X - U)$. Then D is a G-dense set. In fact $\Psi(D) = \Psi(A \cup (X - U)) = \Psi(A) \cup \Psi(X - U) = \Psi(U) \cup \Psi(X - U) = \Psi(U \cup (X - U)) = \Psi(X) = X$. Therefore, $A = D \cap G$, D is a G-dense set and U is a G-regular open set.

(3) $\Rightarrow$ (4): Every G-regular open set is open.

(4) $\Rightarrow$ (1): Suppose $A = U \cap D$ with U and D G-dense. Then $\Psi(A) = \Psi(U)$ since $A = U \cap D$, $\Psi(A) = \Psi(U \cap D) = \Psi(U)$. Hence $A \subset U \subset \Psi(U) = \Psi(A)$, that is, $A \subset \mathrm{Int}(\Psi(A))$. □

Proposition 2.17. *If A is both regular open and G-preopen set in (X, τ, G), then it is G-regular open.*

Proof. $A \subset \mathrm{Int}(\Psi(A)) = \mathrm{Int}(A \cup \Phi(A)) \subset \mathrm{Int}(\mathrm{Cl}(A)) = A$. □

Remark 2.4. It should be noted that open sets and g-sets are independent and regular open sets and G-regular open sets are also independent. Every G-regular open set is open. Regular openness implies openness and G-regular open sets imply g-sets.

3. DECOMPOSITION OF CONTINUITY

Definition 3.1. A function $f\colon (X,\tau) \to (Y,\sigma)$ is said to be B-continuous [10] if for each open set V in Y, $f^{-1}(V)$ is a B-set in X.

Definition 3.2. A function $f\colon (X,\tau,G) \to (Y,\sigma)$ is said to be G-continuous (resp. $G\Phi$-continuous, Φ-continuous, G-precontinuous) if for each open set V in Y, $f^{-1}(V)$ is a G-set (resp. $G\Phi$-set, Φ-open, G-preopen) in (X,τ,G), respectively.

Proposition 3.1. i) *A B-continuous function is G-continuous.*
ii) *A G-continuous function is $G\Phi$-continuous.*

Example 3.1. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$. If $G = \{\{a\},\{b\},\{a,b\}, \{a,c\},\{b,c\},X\}$, then G is a grill on X such that $\tau - \{\emptyset\} \subset G$ [8]. Let $Y = \{a,b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Define a function $f(a) = f(c) = a$ and $f(b) = b$. Then f is G-continuous, but it is neither B-continuous nor G-precontinuous.

Remark 3.1. G-precontinuous and G-continuous are independent from each other as in the following example.

Example 3.2. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$. If $G = \{\{a\},\{b\},\{a,b\}, \{a,c\},\{b,c\},X\}$, then G is a grill on X such that $\tau - \{\emptyset\} \subset G$ [8]. Let $Y = \{a,b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Define a function $f(a) = f(b) = a$ and $f(c) = b$. Then f is G-precontinuous, but it is not G-continuous. In Example 3.1, f is G-continuous, but it is not G-precontinuous.

We have the following decomposition of continuity inspired by Proposition 2.13.

Proposition 3.2. *A function $f\colon (X,\tau,G) \to (Y,\sigma)$ is continuous if and only if it is both G-precontinuous and G-continuous.*

Proof. It follows from Proposition 2.13. □

Proposition 3.3. *If a function $f\colon (X,\tau,G) \to (Y,\sigma)$ is both Φ-continuous and $G\Phi$-continuous, then f is continuous.*

Proof. It follows from Corollary 2.1. □

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