J. Adv. Math. Studies
Vol. **3**(2010), No. 1, 19-32
http://journal.fairpartners.ro

ON THE STOCHASTIC DIFFERENTIAL EQUALITIES

MIRCEA BODNARIU

Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

Abstract. We give a new sense of stochastic differential equalities. We show that if there exists a stochastic process B which satisfies a set of conditions forming what we called 'the H hypothesis', then B could play the same part as the Brownian motion in stochastic calculus and a series of results known in this calculus can be rigorously proved. This new theory is not a formal generalization of the stochastic calculus, it has distinct basis and it develops itself by means of its own tools.

1. INTRODUCTION

Throughout this work, we consider $(\Omega, \mathcal{K}, P)$ be a probability space, $\mathbb{R}$ the set of real numbers, $B\colon [0,\infty) \times \Omega \to \mathbb{R}^m$, $B = (B_1, \dots, B_m)$, a stochastic process which satisfies a set of conditions forming what we called 'the H hypothesis' (see §2), t the time. Let X, u and $v_1, \dots, v_m$ be stochastic processes. In this paper we give a new meaning to the equalities of the following types:

$$(dt)^2 = 0,\ dB_i(t)dt = 0,\ dtdB_i(t) = 0,\ (dB_i(t))^2 = dt, \quad i = 1, \dots, m,$$
$$dX(t) = u(t)dt + v_1(t)dB_1(t) + \cdots + v_m(t)dB_m(t)$$

and to the more general equalities

$$F(a_1(t), \dots, a_m(t), dX_1(t), \dots, dX_n(t)) = 0,$$

where $a_1, \dots, a_m$, $X_1, \dots, X_n$ are stochastic processes and F is a real function. Unlike other authors (e.g. [1]), we do not make use of the stochastic integral.

Using the H hypothesis and the new meaning given to the above equalities, we prove some results known in Itô and Stratonovich calculus and in the extension of this calculus to integrals which depend on a parameter θ in $[0,1]$. In this context, B plays the same part as the Brownian motion in the standard theory.

We emphasize that the theory developed in this paper is not a formal generalization of the stochastic calculus, it has distinct basis and it develops itself by means of its own tools. Nevertheless we obtain the same results as in the standard theory.

Received: August 6, 2009.
2000 *Mathematics Subject Classification*: 60H99.
Key words and phrases: Stochastic differential θ-equality, B-derivative, temporal derivative.

We consider that this fact is very interesting and that the problem 'why we obtain the same results?' is worth studying profoundly.

Naturally, other problem arises from here. Is it possible to develop a theory which contains both the standard theory of stochastic calculus and the new theory exposed here as some particular cases?

In the following we will expose the H hypothesis and the theory which follows from it, leaving the above mentioned problems open for discussion.

2. THE H HYPOTHESIS

If $t, s \in [0,\infty)$, $t \geq s$, θ, $\sigma \in [0,1]$, $\theta \geq \sigma$ and $x\colon [0,\infty) \to \mathbb{R}^k$, then we denote $\Delta_{ts} = t - s$, $\theta_{ts} = \theta(t-s)+s$, $\sigma_{ts} = \sigma(t-s)+s$, $\Delta_{ts}^{\theta\sigma} = \theta_{ts} - \sigma_{ts}$, $\Delta_{ts}x = x(t) - x(s)$ and $\Delta_{ts}^{\theta\sigma}x = x(\theta_{ts}) - x(\sigma_{ts})$.

Let $(\Omega, \mathcal{K}, P)$ be a probability space. We denote $\infty \times \Omega = [0,\infty) \times \Omega$, $\infty^2 \times \Omega = [0,\infty) \times [0,\infty) \times \Omega$ and we introduce the sets

$$\mathcal{R} = \{r : \infty^2 \times \Omega \to \mathbb{R} \mid \lim_{t \searrow s} r(t,s) = r(s,s)\},\ \mathcal{R}_0 = \{r \in \mathcal{R} \mid r(s,s) = 0\}.$$

If $B\colon \infty \times \Omega \to \mathbb{R}^m$, $B = (B_1, \ldots, B_m)$, where for each $i = 1, \ldots, m$ B_i is a stochastic process with continuous paths, then we define

$$\begin{aligned}\mathcal{M}_B = \{f\colon \infty^2 \times \Omega \to \mathbb{R} \mid\ & f(t,s) = r_0(t,s)\Delta_{ts},\ \text{where } r_0 \in \mathcal{R}_0,\\ &\text{or } f(t,s) = (\Delta_{ts}^{\theta\sigma}B_i)^2 - \Delta_{ts}^{\theta\sigma},\ i = 1,\ldots,m,\ \text{where } 0 \leq \sigma < \theta \leq 1,\\ &\text{or } f(t,s) = \Delta_{ts}^{\varepsilon\delta}B_i\Delta_{ts}^{\varphi\psi}B_i,\ i = 1,\ldots,m,\ \text{where } 0 \leq \psi < \varphi \leq \delta < \varepsilon \leq 1\\ &\text{or } f(t,s) = \Delta_{ts}^{\varepsilon\delta}B_i\Delta_{ts}^{\varphi\psi}B_j,\ i,j \in \{1,\ldots,m\},\ i \neq j,\\ &\qquad\qquad \text{where } \delta < \varepsilon,\ \psi < \varphi,\ \delta,\varepsilon,\psi,\varphi \in [0,1]\}.\end{aligned}$$

Remark 2.1. $\mathcal{M}_B \subset \mathcal{R}_0$.

The H hypothesis. There exists a class of stochastic processes $\mathcal{H}_B$ such that:

a) $\mathcal{M}_B \subset \mathcal{H}_B \subset \mathcal{R}_0$.

b) If r_1, $r_2 \in \mathcal{R}$ and h_1, $h_2 \in \mathcal{H}_B$, then $r_1h_1 + r_2h_2 \in \mathcal{H}_B$.

c) If $h(t,s) = \Delta_{ts}^{\theta\sigma}B_i$, $0 \leq \sigma < \theta \leq 1$, $i \in \{1,\ldots,m\}$, then $h \notin \mathcal{H}_B$.

d) If $h(t,s) = g(t,s)\Delta_{ts}$, where $g \notin \mathcal{R}_0$, then $h \notin \mathcal{H}_B$.

e) If $h \in \mathcal{H}_B$, $\varepsilon, \varphi \in [0,1]$, $\varepsilon > \varphi$ and $H(t,s) = h(\varepsilon_{ts}, \varphi_{ts})$, then $H \in \mathcal{H}_B$.

The Brownian motion satisfies the H hypothesis because in this case we can take

$$\mathcal{H}_B = \{h\colon \infty^2 \times \Omega \to \mathbb{R} \mid h(t,s) = \sum_{i=1}^{k} r_i f_i,\ \text{where } r_i \in \mathcal{R} \text{ and } f_i \in \mathcal{M}_B,\ i = 1,\ldots,k\}.$$

Consequence 2.1. 1) If h_1, $h_2 \in \mathcal{H}_B$, then $h_1h_2 \in \mathcal{H}_B$.

Indeed, if $h_1 \in \mathcal{H}_B$, then $h_1 \in \mathcal{R}_0 \subset \mathcal{R}$ and we can use b).

2) If $h(t,s) = (t-s)^2$ or $h(t,s) = \left(\Delta_{ts}^{\theta\sigma}B_i\right)(t-s)$, $0 \leq \sigma < \theta \leq 1$, $i \in \{1,\ldots,m\}$, $i \neq j$, then $h \in \mathcal{H}_B$.

It is true because we can take $r_0(t,s) = t - s$ or $r_0(t,s) = \Delta_{ts}^{\theta\sigma}B_i$ and f belong to $\mathcal{H}_B$ if $f(t,s) = r_0(t,s)\Delta_{ts}$.

3) If $h(t,s) = r_0(t,s)(\Delta_{ts}^{\theta\sigma} B_i)^2$, $i \in \{1,\dots,m\}$, where $r_0 \in \mathcal{R}_0$ and $0 \le \sigma < \theta \le 1$, then $h \in \mathcal{H}_B$.

This is true because $h(t,s) = r_0(t,s)(\theta-\sigma)\Delta_{ts} + r_0(t,s)((\Delta_{ts}^{\theta\sigma} B_i)^2 - \Delta_{ts}^{\theta\sigma})$.

4) If $h(t,s) = \left(\Delta_{ts}^{\theta\sigma} B_i\right)^2$, $0 \le \sigma < \theta \le 1$, $i \in \{1,\dots,m\}$, then $h \notin \mathcal{H}_B$.

To prove this we write $h(t,s) = \left[\left(\Delta_{ts}^{\theta\sigma} B_i\right)^2 - \Delta_{ts}^{\theta\sigma}\right] + \Delta_{ts}^{\theta\sigma}$. If $h \in \mathcal{H}_B$, then $h' \in \mathcal{H}_B$, where $h'(t,s) = h(t,s) - \left[\left(\Delta_{ts}^{\theta\sigma} B_i\right)^2 - \Delta_{ts}^{\theta\sigma}\right] = \Delta_{ts}^{\theta\sigma}$. It results that $h'' \in \mathcal{H}_B$, where $h''(t,s) = \dfrac{1}{\theta-\sigma} h'(t,s) = t-s$, but this is contradictory to d).

We denote $\mathcal{H}_B^n = \{(f_1,\dots,f_n) \mid f_1,\dots,f_n \in \mathcal{H}_B\}$. If $f,g\colon \infty^2 \times \Omega \to \mathbb{R}^n$ and $f-g$ belongs to $\mathcal{H}_B^n$, then we write $f \overset{B}{\equiv} g$. We see that $h \in \mathcal{H}_B^n$ iff $h \overset{B}{\equiv} 0$.

3. THE DEFINITION OF STOCHASTIC DIFFERENTIAL θ-EQUALITIES

Let $B\colon \infty \times \Omega \to \mathbb{R}^n$ be a stochastic process which satisfies the H hypothesis.

Definition 3.1. Let $F\colon \mathbb{R}^p \times \mathbb{R}^{q_1} \times \cdots \times \mathbb{R}^{q_k} \to \mathbb{R}^n$ be a function, $\theta \in [0,1]$ and let $a_i\colon \infty \times \Omega \to \mathbb{R}$, $i = 1,\dots,p$, $X_j\colon \infty \times \Omega \to \mathbb{R}^{q_j}$, $j = 1,\dots,k$, be stochastic processes. If $h_\theta \overset{B}{\equiv} 0$, where

$$h_\theta\colon \infty^2 \times \Omega \to \mathbb{R}^n, \quad h_\theta(t,s) = F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k),$$

then we say that

$$F(a_1,\dots,a_p,dX_1,\dots,dX_k) \overset{\theta}{=} 0.$$

We call the equality above *stochastic differential θ-equality.*

If $\theta = 0$, we say *Itô stochastic differential equality*and we write

$$F(a_1,\dots,a_p,dX_1,\dots,dX_k) \overset{I}{=} 0.$$

If $\theta = \dfrac{1}{2}$, we say *Stratonovich stochastic differential equality*and we write

$$F(a_1,\dots,a_p,dX_1,\dots,dX_k) \overset{S}{=} 0.$$

Consequence 3.1. 1) Let $F\colon \mathbb{R} \to \mathbb{R}$, $F(x) = x^2$. We take $p = 0$, $k = 1$, $q_1 = 1$, $n = 1$, $X_1(t) = X(t) = t$ in Definition 3.1. Then $h_\theta(t,s) = F(t-s) = (t-s)^2$ and, as we have noticed, $h_\theta \overset{B}{\equiv} 0$. It follows that for each $\theta \in [0,1]$ the stochastic differential equality $F(dX) \overset{\theta}{=} 0$ holds, that is $(dX)^2 \overset{\theta}{=} 0$. We will write it

$$(dt)^2 \overset{\theta}{=} 0.$$

2) Let $F\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$, $F(x,y) = xy$, $x = (x_1,\dots,x_m)$. In Definition 3.1 we take $p = 0$, $k = 2$, $q_1 = m$, $q_2 = 1$, $n = m$, $X_1(t) = B(t)$, $X_2(t) = t$. Then $h_\theta(t,s) = F(\Delta_{ts}X_1,\Delta_{ts}X_2) = F(\Delta_{ts}B,\Delta_{ts}) = \Delta_{ts}B\Delta_{ts}$ and $h_\theta \overset{B}{\equiv} 0$. It follows that

for each $\theta \in [0,1]$ the stochastic differential equality $F(dX_1, dX_2) \stackrel{\theta}{=} 0$ holds, i.e. $dX_1 dX_2 \stackrel{\theta}{=} 0$. We write it

$$dBdt \stackrel{\theta}{=} 0 \quad \text{or} \quad dtdB \stackrel{\theta}{=} 0.$$

We can also write for each $i \in \{1, \dots, m\}$

$$dB_i dt \stackrel{\theta}{=} 0 \quad \text{or} \quad dtdB_i \stackrel{\theta}{=} 0.$$

3) Let $p = 0$, $k = 2$, $q_1 = m$, $q_2 = 1$, $n = m^2$, $F\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{m^2}$, $F(x,t) = x \cdot x^T - I_m t$, where $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ is identified with a column matrix, $x^T = (x_1 \dots x_m)$ and I_m is the m-dimensional identity matrix. We identify each element of $\mathbb{R}^{m^2}$ with a matrix from $\mathcal{M}_n(\mathbb{R})$.

If we take $X_1 = B$, $X_2 = t$, then

$$h_\theta(t,s) = F(\Delta_{ts}B, \Delta_{ts}) = \Delta_{ts}B \cdot (\Delta_{ts}B)^T - I_m \Delta_{ts} = (b_{ij}(t,s))_{1 \le i,j \le m},$$

where $b_{ii}(t,s) = (\Delta_{ts}B_i)^2 - \Delta_{ts}$ and $b_{ij}(t,s) = \Delta_{ts}B_i \Delta_{ts}B_j$, $i \neq j$, hence $h_\theta \stackrel{B}{\equiv} 0$. So we have

$$dB\,(dB)^T \stackrel{\theta}{=} I_m dt.$$

It follows that

$$(dB_i)^2 \stackrel{\theta}{=} dt,\ i = 1, \dots, m, \quad dB_i dB_j \stackrel{\theta}{=} 0,\ i \neq j,\ i,j \in \{1, \dots, m\}$$

and

$$(dB)^T dB \stackrel{\theta}{=} mdt.$$

4) The equality $dt \stackrel{\theta}{=} 0$ is false because $h_\theta(t,s) = t - s$ does not belong to $\mathcal{H}_B$.

5) The equality $dB_i \stackrel{\theta}{=} 0$, $i = 1, \dots, m$, is false. Indeed, the stochastic process $h_\theta(t,s) = \Delta_{ts}B_i$ does not belong to $\mathcal{H}_B$.

6) The equality $(dB_i)^2 \stackrel{\theta}{=} 0$, $i = 1, \dots, m$, is false because the stochastic process $h_\theta(t,s) = (\Delta_{ts}B_i)^2$ does not belong to $\mathcal{H}_B$.

Example 3.1. Let $Y\colon \infty \times \Omega \to \mathbb{R}^3$, $Y = \left(\frac{1}{2}B_1^2 + \frac{1}{2}B_2^2,\ B_1B_2,\ B_1 + B_2\right)$. Then

$$dY = \begin{pmatrix} 1 - 2\theta \\ 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \\ 1 & 1 \end{pmatrix} dB.$$

Indeed, if we take $p = 2$, $k = 3$, $q_1 = 1$, $q_2 = 3$, $q_3 = 2$, $n = 3$, $a_1 = B_1$, $a_2 = B_2$, $X_1(t) = t$, $X_2 = Y$, $X_3 = B = (B_1, B_2)$, $F\colon \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$,

$$F(x_1,x_2,t,y_1,y_2,y_3,z_1,z_2)=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}-\begin{pmatrix}1-2\theta\\0\\0\end{pmatrix}t-\begin{pmatrix}x_1&x_2\\x_2&x_1\\1&1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}$$
$$=\begin{pmatrix}y_1-(1-2\theta)\,t-x_1z_1-x_2z_2\\y_2-x_2z_1-x_1z_2\\y_3-z_1-z_2\end{pmatrix},$$

then $h_\theta(t,s)=F(B_1(\theta_{ts}),B_2(\theta_{ts}),\Delta_{ts},\Delta_{ts}Y,\Delta_{ts}B)=(h_{\theta 1}(t,s),\ h_{\theta 2}(t,s),\ h_{\theta 3}(t,s))$ belongs to $\mathcal{H}_B^3$ because

$$\begin{aligned}h_{\theta 1}(t,s)&=\frac{1}{2}\Delta_{ts}(B_1^2)+\frac{1}{2}\Delta_{ts}(B_2^2)-(1-2\theta)\,\Delta_{ts}-B_1(\theta_{ts})\Delta_{ts}B_1-B_2(\theta_{ts})\Delta_{ts}B_2\\&=\frac{1}{2}\left((\Delta_{ts}B_1)^2-\Delta_{ts}\right)+\frac{1}{2}\left((\Delta_{ts}B_2)^2-\Delta_{ts}\right)-\Delta_{ts}^{1\theta}B_1\Delta_{ts}^{\theta 0}B_1\\&\quad-\Delta_{ts}^{1\theta}B_2\Delta_{ts}^{\theta 0}B_2-\left(\left(\Delta_{ts}^{\theta 0}B_1\right)^2-\Delta_{ts}^{\theta 0}\right)-\left(\left(\Delta_{ts}^{\theta 0}B_2\right)^2-\Delta_{ts}^{\theta 0}\right),\end{aligned}$$

$$\begin{aligned}h_{\theta 2}(t,s)&=\Delta_{ts}(B_1B_2)-B_2(\theta_{ts})\Delta_{ts}B_1-B_1(\theta_{ts})\Delta_{ts}B_2\\&=\Delta_{ts}B_1\Delta_{ts}B_2-\Delta_{ts}B_1\Delta_{ts}^{\theta 0}B_2-\Delta_{ts}^{\theta 0}B_1\Delta_{ts}B_2,\end{aligned}$$

$$h_{\theta 3}(t,s)=\Delta_{ts}(B_1+B_2)-\Delta_{ts}B_2-\Delta_{ts}B_1=0$$

belong to $\mathcal{H}_B$.

Proposition 3.1. *Let $\theta,\sigma\in[0,1]$ and $X,u\colon\infty\times\Omega\to\mathbb{R}^n$, $v\colon\infty\times\Omega\to\mathcal{M}_{n,m}(\mathbb{R})$ be stochastic processes, u,v right-continuous, such that*

$$dX\overset{\theta}{=}udt+vdB.\tag{3.1}$$

a) *X is right-continuous.*

b) *If there exist $z\colon\infty\times\Omega\to\mathbb{R}^n$, $w\colon\infty\times\Omega\to\mathcal{M}_{n,m}(\mathbb{R})$ right-continuous such that*

$$dX\overset{\sigma}{=}zdt+wdB,$$

then

(i) $v=w$; (ii) *if $\theta=\sigma$, then $u=z$.*

c) *Let v^i be the i line of the matrix v. If there exist $\varphi\in[0,1]$ and for each $i\in\{1,\dots,n\}$ there exist $a_i\colon\infty\times\Omega\to\mathbb{R}^m$, $b_i\colon\infty\times\Omega\to\mathcal{M}_m(\mathbb{R})$ right-continuous and such that*

$$d(v^i)^T\overset{\varphi}{=}a_idt+b_idB,\tag{3.2}$$

then

$$dX\overset{\sigma}{=}\widetilde{u}\,dt+vdB,\tag{3.3}$$

where $\widetilde{u}=(\widetilde{u}_1\dots\widetilde{u}_n)^T$, $\widetilde{u}_i=u_i+(\theta-\sigma)trb_i$, $u=(u_1\ \dots\ u_n)^T$.

d) *Let* $\alpha, \beta \in \mathbb{R}$ *and* $Y, x\colon \infty \times \Omega \to \mathbb{R}^n$, $y\colon \infty \times \Omega \to \mathcal{M}_{n,m}(\mathbb{R})$ *be stochastic processes such that*

$$dY \stackrel{\theta}{=} xdt + ydB.$$

Then

$$d\left(\alpha X + \beta Y\right) \stackrel{\theta}{=} \left(\alpha u + \beta x\right) dt + \left(\alpha v + \beta y\right) dB.$$

Proof. a) There exist $h_\theta \in \mathcal{H}_B^n$ such that

$$X(t) = X(s) + u(\theta_{ts})\Delta_{ts} + v(\theta_{ts})\Delta_{ts}B + h_\theta(t,s)$$

and because $\lim\limits_{t\searrow s} h_\theta(t,s) = 0$, X is right-continuous.

b) (i) We have

$$\Delta_{ts}X \stackrel{B}{\equiv} u(\theta_{ts})\Delta_{ts} + v(\theta_{ts})\Delta_{ts}B \text{ and } \Delta_{ts}X \stackrel{B}{\equiv} z(\sigma_{ts})\Delta_{ts} + w(\sigma_{ts})\Delta_{ts}B.$$

From here we obtain

$$\left(v(\theta_{ts}) - w(\sigma_{ts})\right)\Delta_{ts}B \stackrel{B}{\equiv} -\left(u(\theta_{ts}) - z(\sigma_{ts})\right)\Delta_{ts}, \tag{3.4}$$

and then $\left(v(\theta_{ts}) - w(\sigma_{ts})\right)\Delta_{ts}B(\Delta_{ts}B)^T \stackrel{B}{\equiv} -\left(u(\theta_{ts}) - z(\sigma_{ts})\right)\Delta_{ts}(\Delta_{ts}B)^T \stackrel{B}{\equiv} 0$. Consequently

$$\begin{aligned}\left(v(\theta_{ts}) - w(\sigma_{ts})\right) I_m\Delta_{ts} = {} & \left(v(\theta_{ts}) - w(\sigma_{ts})\right)\Delta_{ts}B(\Delta_{ts}B)^T \\ & - \left(v(\theta_{ts}) - w(\sigma_{ts})\right)\left(\Delta_{ts}B(\Delta_{ts}B)^T - I_m\Delta_{ts}\right) \equiv 0\end{aligned}$$

and we obtain that $\left(v(\theta_{ts}) - w(\sigma_{ts})\right)\Delta_{ts} \stackrel{B}{\equiv} 0$, but it is possible only if

$$\lim_{t\searrow s}\left(v(\theta_{ts}) - w(\sigma_{ts})\right) = v(s) - w(s) = 0 \text{ for each } s \in [0,\infty).$$

(ii) If $\theta = \sigma$, taking into consideration that $v = w$, from (3.4) it follows that $\left(u(\theta_{ts}) - z(\theta_{ts})\right)\Delta_{ts}$ is decomposable and it is possible only if

$$\lim_{t\searrow s}\left(u(\theta_{ts}) - z(\theta_{ts})\right) = u(s) - z(s) = 0 \text{ for each } s \in [0,\infty).$$

c) The θ-equality $dX \stackrel{\theta}{=} udt + vdB$ is equivalent to

$$\Delta_{ts}X \stackrel{B}{\equiv} u(\theta_{ts})\Delta_{ts} + v(\theta_{ts})\Delta_{ts}B$$

and then

$$\Delta_{ts}X \stackrel{B}{\equiv} u(\sigma_{ts})\Delta_{ts} + v(\sigma_{ts})\Delta_{ts}B + (u(\theta_{ts}) - u(\sigma_{ts}))\Delta_{ts} + (v(\theta_{ts}) - v(\sigma_{ts}))\Delta_{ts}B$$

or $\Delta_{ts}X \stackrel{B}{\equiv} u(\sigma_{ts})\Delta_{ts} + v(\sigma_{ts})\Delta_{ts}B + \Delta_{ts}^{\theta\sigma}v\Delta_{ts}B$ (since $(u(\theta_{ts}) - u(\sigma_{ts}))\Delta_{ts} \stackrel{B}{\equiv} 0$).

Let us suppose $\theta > \sigma$. If $X = (X_1 \ldots X_n)^T$, then

$$\Delta_{ts}X_i \stackrel{B}{\equiv} u_i(\sigma_{ts})\Delta_{ts} + v^i(\sigma_{ts})\Delta_{ts}B + \Delta_{ts}^{\theta\sigma}v^i\Delta_{ts}B, \quad i = 1,\ldots,n.$$

The relation $d(v^i)^T \overset{\varphi}{=} a_i dt + b_i dB$ is equivalent to

$$\Delta_{ts}(v^i)^T \overset{B}{\equiv} a_i(\varphi_{ts})\Delta_{ts} + b_i(\varphi_{ts})\Delta_{ts}B.$$

Replacing here t by θ_{ts} and s by σ_{ts}, we obtain

$$\Delta_{ts}^{\theta\sigma}(v^i)^T \overset{B}{\equiv} a_i(\xi_{ts})(\theta - \sigma)\Delta_{ts} + b_i(\xi_{ts})\Delta_{ts}^{\theta\sigma}B,$$

where $\xi = \varphi(\theta - \sigma) + \sigma$, $\xi \in [0, 1]$, $\xi > \sigma$.

Now, supposing that $b_i = (b_i^{kl})_{1 \le k, l \le m}$, we can write

$$\begin{aligned}
\Delta_{ts}^{\theta\sigma} v^i \Delta_{ts}B &= (\Delta_{ts}B)^T \Delta_{ts}^{\theta\sigma}(v^i)^T \overset{B}{\equiv} (\Delta_{ts}B)^T a_i(\xi_{ts})(\theta - \sigma)\Delta_{ts} + (\Delta_{ts}B)^T b_i(\xi_{ts})\Delta_{ts}^{\theta\sigma}B \\
&\overset{B}{\equiv} \sum_{l=1}^{m}\sum_{k=1}^{m} \Delta_{ts}B_k b_i^{kl}(\xi_{ts})\Delta_{ts}^{\theta\sigma}B_l \overset{B}{\equiv} \sum_{k=1}^{m} b_i^{kk}(\xi_{ts})\Delta_{ts}B_k\Delta_{ts}^{\theta\sigma}B_k \\
&= \sum_{k=1}^{m} b_i^{kk}(\xi_{ts})(((\Delta_{ts}^{\theta\sigma}B_k)^2 - \Delta_{ts}^{\theta\sigma}) + \Delta_{ts}^{1\theta}B_k\Delta_{ts}^{\theta\sigma}B_k + \Delta_{ts}^{\sigma 0}B_k\Delta_{ts}^{\theta\sigma}B_k) \\
&\quad + \sum_{k=1}^{m} b_i^{kk}(\xi_{ts})(\theta - \sigma)\Delta_{ts} \overset{B}{\equiv} trb_i(\xi_{ts})(\theta - \sigma)\Delta_{ts} = trb_i(\sigma_{ts})(\theta - \sigma)\Delta_{ts} \\
&\quad + (trb_i(\xi_{ts}) - trb_i(\sigma_{ts}))(\theta - \sigma)\Delta_{ts} \overset{B}{\equiv} trb_i(\sigma_{ts})(\theta - \sigma)\Delta_{ts}.
\end{aligned}$$

It follows that

$$\Delta_{ts}X_i \equiv (u_i(\sigma_{ts}) + (\theta - \sigma)trb_i(\sigma_{ts}))\,\Delta_{ts} + v^i(\sigma_{ts})\Delta_{ts}B, \quad i = 1, \dots, n,$$

or

$$dX_i \overset{\sigma}{=} (u_i + (\theta - \sigma)trb_i) + v^i dB, \quad i = 1, \dots, n$$

and this is

$$dX \overset{\sigma}{=} \widetilde{u}\,dt + vdB.$$

For $\theta < \sigma$ the proof is similar.

d) From

$$\Delta_{ts}X \overset{B}{\equiv} u(\theta_{ts})\Delta_{ts} + v(\theta_{ts})\Delta_{ts}B \text{ and } \Delta_{ts}Y \overset{B}{\equiv} x(\theta_{ts})\Delta_{ts} + y(\theta_{ts})\Delta_{ts}B$$

we obtain

$$\Delta_{ts}(\alpha X + \beta Y) \overset{B}{\equiv} (\alpha u + \beta x)(\theta_{ts})\Delta_{ts} + (\alpha v + \beta y)(\theta_{ts})\Delta_{ts}B.$$

□

Consequence 3.2. If $X\colon \infty \times \Omega \to R$ and there exist right-continuous functions $u, v\colon \infty \times \Omega \to R$ such that $dX \overset{I}{=} udt + vdB$ and right-continuous functions $a, b\colon \infty \times \Omega \to R$ such that $dv \overset{I}{=} adt + bdB$, then

$$dX \overset{S}{=} \left(u - \frac{1}{2}b\right)dt + vdB.$$

Definition 3.2. Let $\theta \in [0,1]$ and $X\colon \infty \times \Omega \to \mathbb{R}^n$ be a stochastic process. If there exist right-continuous functions $u\colon \infty \times \Omega \to \mathbb{R}^n$, $v\colon \infty \times \Omega \to M_{n,m}(\mathbb{R})$ such that

$$dX \stackrel{\theta}{=} udt + vdB,$$

then we say that X is *θ-differentiable.* We call u the *temporal θ-derivative* of X and we denote it by $D_t^\theta X$ and v the *B-derivative* of X and we denote it by $D_B X$ (see Proposition 3.4).

If $\theta = 0$ we say that X is *Itô-differentiable* and we denote the temporal derivative by $D_t^I X$. If $\theta = \frac{1}{2}$ we say that X is *Stratonovich-differentiable* and we denote the temporal derivative by $D_t^S X$.

When we say that $D_B X$ *exists* (where $X\colon \infty \times \Omega \to \mathbb{R}^n$ is a stochastic process), we mean that there is $\theta \in [0,1]$ such that X is θ-differentiable.

If X is θ-differentiable for each $\theta \in [0,1]$, then we say that X is *differentiable.*

Now we can write the relation (3.1) as

$$dX \stackrel{\theta}{=} D_t^\theta X dt + D_B X dB.$$

If $X = (X_1 \, \ldots \, X_n)^T$, then we denote $D_B X = (D_{B_j} X_i)_{1\le i\le n, 1\le j\le m}$. We see that

$$dX_i \stackrel{\theta}{=} D_t^\theta X_i dt + D_{B_1} X_i dB_1 + \cdots + \; D_{B_m} X_i \, dB_m.$$

We call $D_{B_1} X_i, \ldots, D_{B_m} X_i$ *the partial B-derivatives* of X_i.

Because $D_B X_i = (D_{B_1} X_i \, \ldots \, D_{B_m} X_i)$, b_i from the relation (3.2) becomes $D_B\left(D_B X_i\right)^T = (D^2_{B_j B_k} X_i)_{1\le j,k\le m}$, where $D^2_{B_j B_k} X_i = D_{B_j}(D_{B_k} X_i)$. We denote $\triangle_B X_i = \sum_{j=1}^{m} D^2_{B_j^2} X_i$, $D^2_{B_j^2} X_i = D^2_{B_j B_j} X_i$, and $\triangle_B X = (\triangle_B X_1 \, \ldots \, \triangle_B X_n)^T$. We call $\triangle_B X$ *the B-Laplacian* of X. We see that $\operatorname{tr} b_i = \operatorname{tr}(D_B\left(D_B X_i\right)^T) = \triangle_B X_i$ and then, from relation (3.3) we have

$$D_t^\sigma X = D_t^\theta X + (\theta - \sigma)\triangle_B X. \tag{3.5}$$

In particular $D_t^S X = D_t^I X - \frac{1}{2}\triangle_B X$.

If $X\colon \infty \times \Omega \to \mathbb{R}$, then $\triangle_B X = D_B\left(D_B X\right)$ and we denote it $D^2_{B^2} X$.

4. TRANSFORMATION OF STOCHASTIC DIFFERENTIAL θ-EQUALITIES

Lemma 4.1. *Let $\theta, \sigma, \lambda, \varphi \in [0,1]$ and let $X, Y, Z\colon \infty \times \Omega \to \mathbb{R}^n$ be stochastic processes such that X is θ-differentiable, Y is σ-differentiable and Z is λ-differentiable. Then:*

$$\text{a) } dX\left(dY\right)^T \stackrel{\varphi}{=} \left(D_B X\right)\left(D_B Y\right)^T dt; \tag{4.1}$$

$$\text{b) } dX\left(dY\right)^T dZ \stackrel{\varphi}{=} 0. \tag{4.2}$$

Proof. a) Because X is θ-differentiable and Y is σ-differentiable, there exist right-continuous functions $u, z\colon \infty \times \Omega \to \mathcal{M}_{n,1}(\mathbb{R})$, $v, w\colon \infty \times \Omega \to \mathcal{M}_{n,m}(\mathbb{R})$ such that $dX \overset{\theta}{=} udt + vdB$ and $dY \overset{\sigma}{=} zdt + wdB$. This means that

$$\Delta_{ts}X \overset{B}{\equiv} u(\theta_{ts})\Delta_{ts} + v(\theta_{ts})\Delta_{ts}B \text{ and } \Delta_{ts}Y \overset{B}{\equiv} z(\sigma_{ts})\Delta_{ts} + w(\sigma_{ts})\Delta_{ts}B.$$

So we have

$$\begin{aligned}\Delta_{ts}X(\Delta_{ts}Y)^T &\overset{B}{\equiv} v(\theta_{ts})\Delta_{ts}B\,(\Delta_{ts}B)^T\,(w(\sigma_{ts}))^T \overset{B}{\equiv} v(\theta_{ts})\,(I_m\Delta_{ts})\,(w(\sigma_{ts}))^T\\ &= v(\theta_{ts})\,(w(\sigma_{ts}))^T\,\Delta_{ts} = v(\varphi_{ts})\,(w(\varphi_{ts}))^T\,\Delta_{ts}\\ &\quad + \left[v(\varphi_{ts})\,(w(\sigma_{ts}) - w(\varphi_{ts}))^T + (v(\theta_{ts}) - v(\varphi_{ts}))\,(w(\sigma_{ts}))^T\right]\Delta_{ts}\\ &\overset{B}{\equiv} v(\varphi_{ts})\,(w(\varphi_{ts}))^T\,\Delta_{ts}.\end{aligned}$$

But $v = D_BX$, $w = D_BY$ and we get (4.1).

b) Using the proof of a) we can write

$$\Delta_{ts}X(\Delta_{ts}Y)^T\Delta_{ts}Z \overset{B}{\equiv} \left(v(\varphi_{ts})\,(w(\varphi_{ts}))^T\,\Delta_{ts}Z\right)\Delta_{ts} \overset{B}{\equiv} 0$$

and this is (4.2). □

Proposition 4.1. *Let $F\colon \mathbb{R}^{p+k} \to \mathbb{R}$ be a function and let $a_i\colon \infty \times \Omega \to \mathbb{R}$, $i = 1, \ldots, p$, $X_j\colon \infty \times \Omega \to \mathbb{R}$, $j = 1, \ldots, k$, be stochastic processes.*

a) *If $X, Y, Z\colon \infty \times \Omega \to \mathbb{R}$ are θ-differentiable stochastic processes and $a, b\colon \infty \times \Omega \to \mathbb{R}$ such that*

$$F(a_1, \ldots, a_p, dX_1, \ldots, dX_k) + adX + bdYdZ \overset{\theta}{=} 0, \tag{4.3}$$

then

$$F(a_1, \ldots, a_p, dX_1, \ldots, dX_k) + (aD_t^{\theta}X + bD_BY(D_BZ)^T dt + aD_BXdB \overset{\theta}{=} 0. \tag{4.4}$$

b) *Let $a, X, Y\colon \infty \times \Omega \to \mathbb{R}$ be stochastic processes such that*

$$F(a_1, \ldots, a_p, dX_1, \ldots, dX_k) + adXdY \overset{\theta}{=} 0. \tag{4.5}$$

If

(i) *there are $\varphi \in [0, 1]$ and u_i, $X_i^1\colon \infty\times\Omega \to \mathbb{R}$, $i = 1, \ldots, q$, $X_j^2, X_j^3\colon \infty\times\Omega \to \mathbb{R}$, $j = 1, \ldots, r$, $v_{kl}\colon \infty \times \Omega \to \mathbb{R}$, $k, l \in \{1, \ldots, r\}$ such that*

$$dX \overset{\varphi}{=} \sum_{i=1}^{q} u_i dX_i^1 + \sum_{i,j=1}^{r} v_{ij} dX_i^2 dX_j^3; \tag{4.6}$$

(ii) *there are σ, σ', $\sigma'' \in [0, 1]$ such that Y is σ-differentiable, X_j^2 is σ'-differentiable and X_j^3 is σ''-differentiable for each $j = 1, \ldots, r$;*

then

$$F(a_1, \ldots, a_p, dX_1, \ldots, dX_k) + a\sum_{j=1}^{q} u_i dX_j^1 dY \overset{\theta}{=} 0. \tag{4.7}$$

Proof. a) (4.3) is equivalent to

$$F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)+a(\theta_{ts})\Delta_{ts}X+b(\theta_{ts})\Delta_{ts}Y\Delta_{ts}Z\overset{B}{\equiv}0.$$

Taking into consideration that X, Y, Z are θ-differentiable and using Lemma 4.1, a), we get

$$\begin{aligned}F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)&\\ +[a(\theta_{ts})D_t^{\theta}X(\theta_{ts})+b(\theta_{ts})\left(D_BY\right)(\theta_{ts})\left(D_BZ\right)^T(\theta_{ts})]\Delta_{ts}&\\ +a(\theta_{ts})\left(D_BX\right)(\theta_{ts})\Delta_{ts}B\overset{B}{\equiv}0&\end{aligned}$$

and this is equivalent to (4.4).

b) (4.5) is equivalent to

$$F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)+a(\theta_{ts})\Delta_{ts}X\Delta_{ts}Y\overset{B}{\equiv}0$$

and (4.6) is equivalent to

$$\Delta_{ts}X\overset{B}{\equiv}\sum_{i=1}^{q}u_i(\varphi_{ts})\Delta_{ts}X_i^1+\sum_{i,j=1}^{r}v_{ij}(\varphi_{ts})\Delta_{ts}X_i^2\Delta_{ts}X_j^3.$$

It follows that

$$\begin{aligned}&F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)\\ &\quad+a(\theta_{ts})\sum_{i=1}^{q}u_i(\varphi_{ts})\Delta_{ts}X_i^1\Delta_{ts}Y+a(\theta_{ts})\sum_{i,j=1}^{r}v_{ij}(\varphi_{ts})\Delta_{ts}X_i^2\Delta_{ts}X_j^3\Delta_{ts}Y\overset{B}{\equiv}0.\end{aligned}$$

Because $\Delta_{ts}X_i^2\Delta_{ts}X_j^3\Delta_{ts}Y\overset{B}{\equiv}0$, we have

$$F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)+a(\theta_{ts})\sum_{i=1}^{q}u_i(\varphi_{ts})\Delta_{ts}X_i^1\Delta_{ts}Y\overset{B}{\equiv}0.$$

Then

$$\begin{aligned}&F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)\\ &\quad+a(\theta_{ts})\sum_{i=1}^{q}u_i(\theta_{ts})\Delta_{ts}X_i^1\Delta_{ts}Y+a(\theta_{ts})\sum_{i=1}^{q}\left(u_i(\varphi_{ts})-u_i(\theta_{ts})\right)\Delta_{ts}X_i^1\Delta_{ts}Y\overset{B}{\equiv}0.\end{aligned}$$

But $\Delta_{ts}X_i^1\Delta_{ts}Y\overset{B}{\equiv}\left(D_BX_i^1\right)(\varphi_{ts})\left(D_BY\right)^T(\varphi_{ts})\Delta_{ts}$ (Lemma 3.1 a)) and we obtain $a(\theta_{ts})\sum_{i=1}^{q}\left(u_i(\varphi_{ts})-u_i(\theta_{ts})\right)\Delta_{ts}X_i^1\Delta_{ts}Y\overset{B}{\equiv}0$. Finally,

$$F(a_1(\theta_{ts}),\dots,a_p(\theta_{ts}),\Delta_{ts}X_1,\dots,\Delta_{ts}X_k)+a(\theta_{ts})\sum_{i=1}^{q}u_i(\theta_{ts})\Delta_{ts}X_i^1\Delta_{ts}Y\overset{B}{\equiv}0$$

and this is equivalent to (4.7). □

Proposition 4.2. *Let $\theta, \sigma, \varphi \in [0,1]$ and $a_i, X_i, b_j, Y_j, Z_j \colon \infty \times \Omega \to \mathbb{R}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, be φ-differentiable stochastic processes and $b_j \colon \infty \times \Omega \to \mathbb{R}$, $j = 1, \ldots, q$, be right-continuous stochastic process. Then*

$$\sum_{i=1}^{p} a_i dX_i + \sum_{j=1}^{q} b_j dY_j dZ_j - \theta \sum_{i=1}^{p} da_i dX_i \overset{\theta}{=} 0 \tag{4.8}$$

if and only if

$$\sum_{i=1}^{p} a_i dX_i + \sum_{j=1}^{q} b_j dY_j dZ_j - \sigma \sum_{i=1}^{p} da_i dX_i \overset{\sigma}{=} 0. \tag{4.9}$$

Proof. (4.8) is equivalent to

$$\sum_{i=1}^{p} a_i(\theta_{ts})\Delta_{ts}X_i + \sum_{j=1}^{q} b_j(\theta_{ts})\Delta_{ts}Y_j\Delta_{ts}Z_j - \theta \sum_{i=1}^{p} \Delta_{ts}a_i\Delta_{ts}X_i \overset{B}{\equiv} 0$$

and (4.9) is equivalent to

$$\sum_{i=1}^{p} a_i(\sigma_{ts})\Delta_{ts}X_i + \sum_{j=1}^{q} b_j(\sigma_{ts})\Delta_{ts}Y_j\Delta_{ts}Z_j - \sigma \sum_{i=1}^{p} \Delta_{ts}a_i\Delta_{ts}X_i \overset{B}{\equiv} 0.$$

It is sufficient to prove that

$$\sum_{i=1}^{p} \Big(\Delta_{ts}^{\theta\sigma}a_i - (\theta - \sigma)\,\Delta_{ts}a_i\Big)\,\Delta_{ts}X_i + \sum_{j=1}^{q} \Delta_{ts}^{\theta\sigma}b_j\Delta_{ts}Y_j\Delta_{ts}Z_j \overset{B}{\equiv} 0.$$

From Lemma 4.1 a) we get $\Delta_{ts}Y_j\Delta_{ts}Z_j \overset{B}{\equiv} (D_BY_j)\,(\varphi_{ts})\,(D_BZ_j)^T\,(\varphi_{ts})\Delta_{ts}$ and then $\sum_{j=1}^{q} \Delta_{ts}^{\theta\sigma}b_j\Delta_{ts}Y_j\Delta_{ts}Z_j \overset{B}{\equiv} 0$. It remains to prove that

$$\sum_{i=1}^{p} \Big(\Delta_{ts}^{\theta\sigma}a_i - (\theta - \sigma)\,\Delta_{ts}a_i\Big)\,\Delta_{ts}X_i \overset{B}{\equiv} 0.$$

We will prove that $\big(\Delta_{ts}^{\theta\sigma}a - (\theta-\sigma)\,\Delta_{ts}a\big)\,\Delta_{ts}X \overset{B}{\equiv} 0$, where $a, X \colon \infty \times \Omega \to \mathbb{R}$ are φ-differentiable.

We have $\Delta_{ts}a \overset{B}{\equiv} (D_t^{\varphi}a)\,(\varphi_{ts})\Delta_{ts} + \sum_{k=1}^{m} (D_{B_k}a)\,(\varphi_{ts})\Delta_{ts}B_k$. Replacing here t by θ_{ts} and s by σ_{ts}, we obtain

$$\Delta_{ts}^{\theta\sigma}a \overset{B}{\equiv} (D_t^{\varphi}a)\,(\xi_{ts})(\theta-\sigma)\Delta_{ts} + \sum_{k=1}^{m} (D_{B_k}a)\,(\xi_{ts})\Delta_{ts}^{\theta\sigma}B_k,$$

where $\xi = \varphi(\theta - \sigma) + \sigma$. Now we have

$$\Delta_{ts}^{\theta\sigma} a - (\theta - \sigma)\,\Delta_{ts} a \overset{B}{\equiv} \left((D_t^{\varphi} a)(\xi_{ts}) - (D_t^{\varphi} a)(\varphi_{ts})\right)(\theta - \sigma)\Delta_{ts} + \sum_{k=1}^{m} \Big((D_{B_k} a)(\xi_{ts})\Delta_{ts}^{\theta\sigma} B_k - (\theta - \sigma)(D_{B_k} a)(\varphi_{ts})\Delta_{ts} B_k\Big).$$

On the other hand $\Delta_{ts} X \overset{B}{\equiv} (D_t^{\varphi} X)(\varphi_{ts})\Delta_{ts} + \sum_{k=1}^{m} (D_{B_k} X)(\varphi_{ts})\Delta_{ts} B_k$.

Now we can write

$$\begin{aligned}
&\Big(\Delta_{ts}^{\theta\sigma} a - (\theta - \sigma)\,\Delta_{ts} a\Big)\Delta_{ts} X \overset{B}{\equiv} \left((D_t^{\varphi} a)(\xi_{ts}) - (D_t^{\varphi} a)(\varphi_{ts})\right)(\theta - \sigma)\Delta_{ts} X \Delta_{ts} \\
&\quad + \sum_{k=1}^{m} \Big((D_{B_k} a)(\xi_{ts})\Delta_{ts}^{\theta\sigma} B_k - (\theta - \sigma)(D_{B_k} a)(\varphi_{ts})\Delta_{ts} B_k\Big)\Delta_{ts} X \\
&\quad \overset{B}{\equiv} \sum_{k=1}^{m} \Big((D_{B_k} a)(\xi_{ts})\Delta_{ts}^{\theta\sigma} B_k - (\theta - \sigma)(D_{B_k} a)(\varphi_{ts})\Delta_{ts} B_k\Big)(D_t^{\varphi} X)(\varphi_{ts})\Delta_{ts} \\
&\quad + \sum_{k=1}^{m}\sum_{l=1}^{m} \Big((D_{B_k} a)(\xi_{ts})\Delta_{ts}^{\theta\sigma} B_k - (\theta - \sigma)(D_{B_k} a)(\varphi_{ts})\Delta_{ts} B_k\Big)(D_{B_l} X)(\varphi_{ts})\Delta_{ts} B_l \\
&\quad \overset{B}{\equiv} \sum_{k=1}^{m} \Big((D_{B_k} a)(\xi_{ts})\Delta_{ts}^{\theta\sigma} B_k \Delta_{ts} B_k - (\theta - \sigma)(D_{B_k} a)(\varphi_{ts})(\Delta_{ts} B_k)^2\Big)(D_{B_k} X)(\varphi_{ts}) \\
&\quad \overset{B}{\equiv} \sum_{k=1}^{m} (D_{B_k} X)(\varphi_{ts}) \left((D_{B_k} a)(\xi_{ts})\left(\Delta_{ts}^{\theta\sigma} B_k\right)^2 - (\theta - \sigma)(D_{B_k} a)(\varphi_{ts})\Delta_{ts}\right) \\
&\quad \overset{B}{\equiv} \sum_{k=1}^{m} (D_{B_k} X)(\varphi_{ts})\left((D_{B_k} a)(\xi_{ts}) - (D_{B_k} a)(\varphi_{ts})\right)(\theta - \sigma)\,\Delta_{ts} \overset{B}{\equiv} 0.
\end{aligned}$$

□

5. ITÔ FORMULA AND θ-FORMULA

Theorem 5.1 (Itô formula). *Let $g\colon \mathbb{R}^n \to \mathbb{R}$, $g \in C^2(\mathbb{R}^n)$ and let $X\colon \infty \times \Omega \to \mathbb{R}^n$, $X = (X_1, \dots, X_n)$, be Itô-differentiable stochastic process.*

Then the following stochastic differential equalities hold:

$$\text{a) } dg(X) \overset{I}{=} \nabla g(X) dX + \frac{1}{2}(dX)^T H(g)(X) dX, \tag{5.1}$$

where $\nabla g(X) = \left(\frac{\partial g}{\partial x_1}(X) \cdots \frac{\partial g}{\partial x_n}(X)\right)$, $H(g)(X) = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(X)\right)_{1\le i,j\le n}$.

$$\text{b) } dg(X) \overset{I}{=} \left(\nabla g(X) D_t^I X + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X)(D_B X_i)(D_B X_j)^T\right) dt + \nabla g(X) D_B X dB. \tag{5.2}$$

Proof. a) We have to show that

$$\Delta_{ts}g(X) - \nabla g(X(s))\Delta_{ts}X - \frac{1}{2}(\Delta_{ts}X)^T H(g)(X(s))(\Delta_{ts}X) \overset{B}{\equiv} 0.$$

Using Taylor formula the above relation is equivalent to

$$\omega_s(X(t))\left[\sum_{i=1}^{n}(\Delta_{ts}X_i)^2\right] \overset{B}{\equiv} 0, \tag{5.3}$$

where ω_s is continuous and $\lim\limits_{x\to X(s)} \omega_s(x) = 0$. X is Itô-differentiable, hence X is right-continuous and then $\lim\limits_{t\searrow s}\omega_s(X(t)) = 0$.

Because X is Itô-differentiable we can write $(\Delta_{ts}X_i)^2 \overset{B}{\equiv} D_BX_i(s)\,(D_BX_i)^T\,(s)\Delta_{ts}$ (Lemma 4.1) and (5.3) becomes equivalent to

$$\sum_{i=1}^{n}\omega_s(X(t))D_BX_i(s)\cdot(D_BX_i)^T\,(s)\Delta_{ts} \overset{B}{\equiv} 0$$

which is true because $\lim\limits_{t\searrow s}\omega_s(X(t))D_BX_i(s)\cdot(D_BX_i)^T\,(s) = 0$.

b) At a) we have proved that

$$dg(X) - \sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(X)dX_i - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 g}{\partial x_i\partial x_j}(X)dX_idX_j \overset{I}{=} 0.$$

Applying Proposition 4.2 we get

$$dg(X) - \left(\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(X)D_t^IX_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 g}{\partial x_i\partial x_j}(X)D_BX_i\cdot(D_BX_j)^T\right)dt$$
$$-\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(X)\,(D_BX_i)\,dB \overset{I}{=} 0$$

which is (5.2). □

Theorem 5.2 (θ-formula)). *Let $\theta \in [0,1]$, $g\colon \mathbb{R}^n \to \mathbb{R}$, $g \in C^3(\mathbb{R}^n)$ and let $X\colon \infty\times\Omega \to \mathbb{R}^n$, $X = (X_1,\ldots,X_n)$, be differentiable stochastic process. Then the following stochastic differential equalities hold:*

$$\text{a) } dg(X) \overset{\theta}{=} \nabla g(X)dX + \left(\frac{1}{2}-\theta\right)(dX)^T\cdot H(g)(X)\cdot dX; \tag{5.4}$$

$$\text{b) } dg(X) \overset{\theta}{=} \left(\nabla g(X)D_t^\theta X + \left(\frac{1}{2}-\theta\right)\sum_{i,j=1}^{n}\frac{\partial^2 g}{\partial x_i\partial x_j}(X)D_BX_i\cdot(D_BX_j)^T\right)dt$$
$$+\nabla g(X)D_BXdB.$$

Proof. a) Because X is differentiable, it is Itô-differentiable and (5.1) is true. Now we can apply Proposition 4.2 and we get

$$dg(X) \overset{\theta}{=} \nabla g(X)dX + \frac{1}{2}(dX)^T \cdot H(g)(X) \cdot dX - \theta \sum_{i=1}^{n} d\left(\frac{\partial g}{\partial x_i}\right)(X)dX_i.$$

Using a) from Theorem 4.1, we obtain

$$d\left(\frac{\partial g}{\partial x_i}\right) \overset{I}{=} \sum_{j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X)dX_j + \frac{1}{2}\sum_{j,l=1}^{n} \frac{\partial^3 g}{\partial x_i \partial x_j \partial x_l}(X)dX_j dX_l.$$

From Proposition 4.1 we get

$$dg(X) \overset{\theta}{=} \nabla g(X)dX + \frac{1}{2}(dX)^T \cdot H(g)(X) \cdot dX - \theta \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(X)dX_j dX_i$$

which is (5.4).

b) The proof is similar to the proof of Theorem 5.1 b). □

Application 5.1. Let $X\colon \infty \times \Omega \to \mathbb{R}^n$ be a differentiable stochastic process, $a\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ continuous, $b\colon [0,\infty) \times \mathbb{R}^n \to \mathcal{M}_{n,m}(\mathbb{R})$, $b = (b_{ij})_{1\le i\le n, 1\le j\le m}$, $b_{ij} \in C^2(\mathbb{R}^n)$. Let b_k^c be the k column of b and $\dfrac{\partial b_k^c}{\partial x_j} = \left(\dfrac{\partial b_{1k}}{\partial x_j} \cdots \dfrac{\partial b_{nk}}{\partial x_j}\right)^T$.

If $dX \overset{\theta}{=} a(t,X)dt + b(t,X)dB$, then

$$dX \overset{\sigma}{=} \left(a(t,X) + (\theta - \sigma)\sum_{k=1}^{m}\sum_{j=1}^{n} \frac{\partial b_k^c}{\partial x_j}(t,X) b_{jk}(t,X)\right) dt + b(t,X)dB.$$

Proof. Itô formula allows us to find $\triangle_B X = \displaystyle\sum_{k=1}^{m}\sum_{j=1}^{n} \frac{\partial b_k^c}{\partial x_j}(t,X) b_{jk}(t,X)$ an then applying (3.5) we get the σ-equality. □

Acknowledgments. The author would like to thank professor Constantin Udrişte for his suggestions and encouraging comments about this paper.

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University "Politehnica" of Bucharest
Faculty of Applied Sciences
Splaiul Independenţei, No. 313, 060042 *Bucharest, Romania*
E-mail address: m.bodnariu@gmail.com