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METRIZABLE LINEAR CONNECTIONS IN A LIE ALGEBROID

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

ABSTRACT. A linear connection D in a Lie algebroid is said to be metrizable if there exists a Riemannian metric h in the Lie algebroid such that $Dh = 0$. Conditions for the linear connection D to be metrizable are investigated.

1. INTRODUCTION

Lie algebroids as particular anchored vector bundles [21] have now an important place in differential geometry and algebraic geometry. Initially defined as infinitesimal part of Lie grupoids, their algebra [12] and geometry is independently and largely developed [5, 8, 11, 20]. Besides they have proved to be useful in Mechanics [2, 4, 7, 16, 24], in the theory of nonholonomic systems [3, 9, 18] in control theory [6], in field theory [16], in quantum and classical gravity [22, 23]. The cohomology of Lie algebroids started in [17]. There the adapted exterior differential d was for the first time introduced. See also [13]. Holonomy and characteristic classes have been studied in [10]. For more references see the monograph [14]

Let A be a Lie algebroid and D an A-connection (defined in the Section 3) in a vector bundle (F, q, M). We say that D is metrizable if there exists a Riemannian metric h in (F, q, M) such that $Dh = 0$. The tangent bundle TM is a trivial Lie algebroid and a TM-connection in (F, q, M) is nothing but an usual linear connection in this vector bundle. In [1] we provided conditions for metrizability of a TM-connection in any vector bundle as well as in vector bundles endowed with Finsler functions. In this paper we extend some results from [1] to A-connection in (F, q, M). The notations from [19] are used.

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2. LIE ALGEBROIDS

Let $\xi = (E, q, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n + m$ and $p\colon E \longrightarrow M$ is a smooth submersion. The fibres $E_x = p^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\mathcal{F}(M)$ be the ring of smooth real functions on M. We denote by $\Gamma(E)$ and $\mathcal{X}(M)$ the $\mathcal{F}(M)$-module of sections of ξ and of the tangent bundle (TM, τ, M), respectively. If $\left(U, \left(x^i\right)\right)$, $i, j, k \ldots = 1, 2, \ldots, n$ is a local chart on M, then $\left(\dfrac{\partial}{\partial x^i}\right)$ provide a local basis for $\mathcal{X}(U)$. Let $s_a\colon U \longrightarrow p^{-1}(U)$, $a, b, c, \ldots = 1, 2, \ldots, m$ be a local basis for $\Gamma\left(p^{-1}(U)\right)$. Any section s over U has the form $s = y^a s_a(x)$, $x \in U$ and we will take $\left(x^i, y^a\right)$ as local coordinates on $p^{-1}(U)$. A change of these coordinates $\left(x^i, y^a\right) \longrightarrow \left(\tilde{x}^i, \tilde{y}^a\right)$ has the form

$$\begin{aligned} &\tilde{x}^i = \tilde{x}^i\left(x^1, \ldots, x^n\right), \quad \operatorname{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \\ &\tilde{y}^a = M^a_b(x) y^b, \qquad \operatorname{rank}\left(M^a_b(x)\right) = m. \end{aligned} \tag{2.1}$$

Let $\xi^* = (E^*, p^*, M)$ be the dual of vector bundle ξ and $\theta^a\colon U \longrightarrow p^{*-1}(U)$, $x \longrightarrow \theta^a(x) \in E^*_x$ a local basis for $\Gamma\left(p^{*-1}(U)\right)$ such that $\theta^a(s_b) = \delta^a_b$.

Next, we may consider the tensor bundle of type (r, s), $\mathcal{T}^r_s(E)$ over M and its sections. For $g \in \Gamma\left(E^* \otimes E^*\right)$ we have $g = g_{ab}(x)\theta^a \otimes \theta^b$. As $(E^* \otimes E^*) \cong L_2(E, \mathbb{R})$, we may regard g as a smooth mapping $x \longrightarrow g(x)\colon E_x \times E_x \longrightarrow \mathbb{R}$ with $g(x)$ a bilinear mapping given by $g(x)(s_a, s_b) = g_{ab}(x)$, $x \in M$.

If the mapping $g(x)$ is symmetric i.e. $g_{ab} = g_{ba}$ and positive definite i.e. $g_{ab}(x)\xi^a\xi^b > 0$ for every $(\xi^a) \neq 0$, one says that g defines a Riemannian metric in the vector bundle ξ.

Let us assume that

(i) $\Gamma(E)$ is endowed with a Lie algebra structure $[,]$ over $\mathbb{R}$,

(ii) There exists a bundle map $\rho\colon E \longrightarrow TM$, called anchor map. It induces a Lie algebra homomorphism (denoted also by ρ) from $\Gamma(E)$ to $\mathcal{X}(M)$,

(iii) For any sections $s_1, s_2 \in \Gamma(E)$ and for any $f \in \mathcal{F}(M)$ the following identity holds

$$[s_1, f s_2] = f[s_1, s_2] + \rho(s_1) f s_2.$$

Definition 2.1. The triplet $A = (\xi, [,], \rho)$ with the properties (i), (ii) and (iii) is called a *Lie algebroid*.

Examples:

1. The tangent bundle (TM, τ, M) with the usual Lie bracket and ρ equal to the identity map form a Lie algebroid.

2. Any integrable subbundle of TM with the Lie bracket defined by restriction and ρ the inclusion map is a Lie algebroid.
3. Let (F, q, M) be any vector bundle. On F we have the vertical distribution $u \longrightarrow V_uF = \text{Ker} q_{*,u}$, $u \in F$, where q_* denotes the differential of q.

This distribution is integrable. If we regard it as a subbundle of TF, accordingly to Example 2 a Lie algebroid is obtained.

Locally, we set

$$\rho(s_a) = \rho_a^i(x)\frac{\partial}{\partial x^i}, \quad [s_a, s_b] = L_{ab}^c(x)s_c. \tag{2.2}$$

The structure function ρ_a^i and L_{ab}^c of the Lie algebroid A have to satisfy the following identities

$$\begin{gathered}\rho_a^i\frac{\partial\rho_b^j}{\partial x^i} - \rho_b^i\frac{\partial\rho_a^j}{\partial x^i} = \rho_c^j L_{ab}^c, \qquad L_{ab}^c + L_{ba}^c = 0,\\ \sum_{cycl(abc)}\left(L_{ab}^d L_{dc}^e + \rho_c^i\frac{\partial L_{ab}^e}{\partial x^i}\right) = 0.\end{gathered} \tag{2.3}$$

3. CONNECTIONS IN LIE ALGEBROIDS

Let $A = (\xi, [,], \rho)$ be a Lie algebroid with $\xi = (E, p, M)$ and let (F, q, M) be any vector bundle.

Definition 3.1. An *A-connection* in the bundle (F, q, M) is a mapping

$$D\colon \Gamma(E) \times \Gamma(F) \longrightarrow \Gamma(F), \quad (s, \sigma) \longrightarrow D_s\sigma$$

with the properties:

1) $D_{s_1+s_2}\sigma = D_{s_1}\sigma + D_{s_2}\sigma$,
2) $D_{fs}\sigma = fD_s\sigma$,
3) $D_s(\sigma_1 + \sigma_2) = D_s\sigma_1 + D_s\sigma_2$,
4) $D_s(f\sigma) = \rho(s)f\sigma + fD_s\sigma$,

for $s, s_1, s_2 \in \Gamma(E)$, $\sigma, \sigma_1, \sigma_2 \in \Gamma(F)$, $f \in \mathcal{F}(M)$.

Notice that a TM-connection in the vector bundle (F, q, M) is nothing but a linear connection in this vector bundle.

Definition 3.2. An A-connection in the bundle $\xi = (E, p, M)$ is called a *linear connection* in the Lie algebroid A.

The notion of tangent lift of a curve on M is generalized as follows.

Definition 3.3. Let $A = (\xi, [,], \rho)$ be a Lie algebroid with $\xi = (E, p, M)$. A curve $\alpha\colon [0, 1] \longrightarrow E$ is called *admissible* or an *A-path* if $\rho(\alpha(t)) = \dfrac{d}{dt}p(\alpha(t))$, $t \in [0, 1]$. The curve $\gamma(t) = p(\alpha(t))$ will be called the *base path* of α. The A-path α is called *vertical* if $\rho(\alpha(t)) = 0$. In this case γ reduces to a point and the curve α is contained in the fibre in that point.

Locally, if $\alpha(t) = \big(x^i(t), y^a(t)\big)$, then $\gamma(t) = (x^i(t))$ and α is an A-path if and only if

$$\rho^i_a(x(t))y^a(t) = \frac{dx^i(t)}{dt}, \qquad t \in [0,1], \tag{3.1}$$

and it is a vertical A-path if and only if

$$\rho^i_a(x(t))y^a(t) = 0, \qquad t \in [0,1]. \tag{3.2}$$

Let (σ_α), $\alpha, \beta, \gamma, \ldots = k{:}=$rank of (F, q, M), a local basis in $\Gamma(F)$. Then a local section σ has the form $\sigma = z^\alpha \sigma_\alpha$ and (z^α) are the coordinates in the fibres of (F, q, M).

For $s = y^a s_a$ and $\sigma = z^\alpha \sigma_\alpha$, by the Definition 3.1 we have

$$D_s\sigma = y^a \left(\rho^i_a \frac{\partial z^\alpha}{\partial x^i} + z^\alpha D_{s_a})\sigma_\alpha\right)$$

and if we put

$$D_{s_a}\sigma_\alpha = \Gamma^\beta_{\alpha a}\sigma_\beta,$$

we get

$$D_s\sigma = y^a(D_a z^\beta)\sigma_\beta, \qquad D_a z^\beta = \rho^i_a \frac{\partial z^\beta}{\partial x^i} + \Gamma^\beta_{\alpha a} z^\alpha. \tag{3.3}$$

For a linear connection $\mathcal{D}$ in the Lie algebroid $A = (\xi, [,], \rho)$ we get

$$\mathcal{D}_s\sigma = y^a(\mathcal{D}_a z^b)s_b, \qquad \mathcal{D}_a z^b = \rho^i_a \frac{\partial z^b}{\partial x^i} + \Gamma^b_{ca} z^c.$$

Let D be an A-connection in the vector bundle (F, q, M) and $\alpha\colon [0,1] \longrightarrow E$ an A-path.

A smooth mapping $\sigma\colon [0,1] \longrightarrow F$ is called an α-section if $q(\sigma(t)) = p(\alpha(t))$, $t \in [0,1]$.

Locally, if $\alpha(t) = (x^i(t), y^a(t))$ then $\sigma(t) = (x^i(t), z^\alpha(t))$.

Let $\Gamma(F)^\alpha$ be the linear space of α-section in the vector bundle (F, q, M). We define an operator $D^\alpha\colon \Gamma(F)^\alpha \longrightarrow \Gamma(F)^\alpha$, $\sigma(t) \longrightarrow (D^\alpha\sigma)(t)$ by

$$(D^\alpha\sigma)(t) = \left(\frac{dz^\beta}{dt} + \Gamma^\beta_{\alpha a}(x(t))z^\alpha(t)y^a(t)\right)\sigma_\beta, \tag{3.4}$$

whenever $\sigma(t) = z^\alpha(t)\sigma_\alpha$.

The operator D^α has the following properties:

(i) $D^\alpha(c_1\sigma_1 + c_2\sigma_2) = c_1 D^\alpha\sigma_1 + c_2 D^\alpha\sigma_2$, $\quad c_1, c_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \Gamma(F)^\alpha$,

(ii) $D^\alpha(fs) = \dfrac{df}{dt}\sigma + f D^\alpha\sigma$, for $\sigma \in \Gamma(F)^\alpha$ and $f\colon [0,1] \longrightarrow \mathbb{R}$ a smooth function,

(iii) If $\widetilde{\sigma}$ is a local section that extends $\sigma \in \Gamma(F)^\alpha$ and $\rho(\alpha(t)) \neq 0$ ($\rho(\alpha(t)) = 0$) then $(D^\alpha\sigma)(t) = D_{\alpha(t)}\widetilde{\sigma}$ (resp. $(D^\alpha\sigma)(t) = D_{\alpha(t)}\widetilde{\sigma} + \dfrac{d\sigma}{dt}$).

The first two properties are immediate by (3.4). To prove (iii) one uses (3.4), (3.2) and (3.3)

By contradiction one proves that D^α is the unique operator with the properties (i), (ii) and (iii). Indeed, if $\widetilde{D}^\alpha\colon \Gamma(F)^\alpha \longrightarrow \Gamma(F)^\alpha$ is another operator satisfying (i), (ii) and (iii), it easily follows that it is has the form that appears in the second hand of (3.4).

Definition 3.4. An α-section σ in the vector bundle (F, q, M) is said to be *parallel* if $D^\alpha\sigma = 0$.

Locally, the α-section $\sigma(t) = z^\alpha(t)\sigma_\alpha$ with $\alpha(t) = y^a(t)s_a$ is parallel if and only if the functions $z^\alpha(t)$ are solutions of the following linear system of ordinary differential equations

$$\frac{dz^\beta}{dt} + \Gamma^\beta_{\alpha a}(x(t))z^\alpha(t)y^a(t) = 0. \tag{3.5}$$

This system has an unique solution $t \longrightarrow \sigma(t)$ with the initial condition $\sigma(0) = \sigma_0$. This fact allow us to define the parallel displacement along α, denoted by

$$P^t_\alpha\colon F_{\gamma(0)} \longrightarrow F_{\gamma(t)}, \quad \gamma(t) = p(\alpha(t) = q(\sigma(t)), \quad P^t_\alpha(\sigma_0) = \sigma(t).$$

The maps P^t_α are linear isomorphisms.

In particular, we may take α a loop based at $x \in M$ i.e. $\gamma(0) = \gamma(1) = x$ and we get the linear isomorphism $P_\alpha\colon F_x \mapsto F_x$. Its inverse is $P_{\alpha^{-1}}$ where α^{-1} is the reverse loop of α and if we consider the composite $\alpha_1 \circ \alpha_2$ that is α_2 followed by α_1 of two loops based on x it comes out that $P_{\alpha_1\circ\alpha_2} = P_{\alpha_2} \circ P_{\alpha_1}$. On this way one obtains a subgroup of the linear isomorphisms of F_x called the holonomy group of D, denoted by $\Phi(x)$).

We fix t and consider $(P^t_\alpha)^{-1}\colon F_{\gamma(t)} \longrightarrow F_{\gamma(0)}$. Locally, if $(P^\tau t_\alpha)^{-1}(\sigma(\tau)) = \widetilde{z}^\beta(\tau)\sigma_\beta$, then $\widetilde{z}^\beta(t) = z^\beta(t)$, $\widetilde{z}^\beta(0) = (P^t_\alpha)^{-1}(\sigma(t))$ and $(\widetilde{z}^\beta)$ are solutions of (3.5). By Taylor's formula $\widetilde{z}^\beta(t) = \widetilde{z}^\beta(0) + t\dfrac{d\widetilde{z}^\beta}{dt}(0)\ldots$, hence

$$z^\beta(t) - z^\beta(0) = ((P^t_\alpha)^{-1}(\sigma(t))^\beta - z^\beta(0) + t\frac{d\widetilde{z}^\beta}{dt}(0)\ldots.$$

We divide this by t, take $t \to 0$ and obtain

$$(D^\alpha\sigma)(0) = \lim_{t\to 0}\frac{(P^t_\alpha)^{-1}(\sigma(t)) - \sigma(0)}{t}.$$

Suppose now that $\mathcal{D}$ is a linear connection in the Lie algebroid $A = (\xi, [,], \rho)$, $\xi = (E, p, M)$. An α-path is called geodesic if $\mathcal{D}^\alpha\alpha = 0$. Locally, if $\alpha(t) = (x^i(t), y^a(t))$, then $\mathcal{D}^\alpha\alpha = \left[\dfrac{dy^a}{dt} + \Gamma^a_{bc}(x(t))y^b(t)y^c(t)\right]s_a$ and α is a geodesic if and only if the functions $x^i(t), y^a(t)$ are solution of the following system of ordinary differential equations:

$$\frac{dx^i}{dt} = \rho_a^i(x(t))y^a(t),$$
$$\frac{dy^a}{dt} + \Gamma_{bc}^a(x(t))y^b(t)y^c(t) = 0.$$

It is clear that one has the existence and uniqueness of geodesic with a given base point $x \in M$ and a given $y_0 \in E_{x_0}$. If for a pair (x_0, y_0) we have $\rho_a^i(x_0)y_0^a = 0$, the corresponding geodesic is contained in the fibre E_{x_0} i.e. it is a vertical A-path.

4. RIEMANNIAN METRICS IN LIE ALGEBROIDS

Let $A = (\xi, [,], \rho)$ be a Lie algebroid with $\xi = (E, p, M)$ and a vector bundle (F, q, M) endowed with an A-connection D whose local coefficients are $(\Gamma_{\beta a}^{\alpha})$.

A Riemannian metric in (F, q, M) is a mapping g that assigns to any $x \in M$ a scalar product g_x in E_x such that for any local section $\sigma_1, \sigma_2 \in \Gamma(F)$, the function $x \longrightarrow g_x(\sigma_1, \sigma_2)$ is smooth. Locally, we set $g_x(\sigma_\alpha, \sigma_\beta) = g_{\alpha\beta}(x)$ and so $g_x(\sigma_1, \sigma_2) = g_{\alpha\beta}(x)z_1^\alpha z_2^\beta$ if $\sigma_1 = z_1^\alpha \sigma_\alpha$, $\sigma_2 = z_2^\beta \sigma_\beta$.

The operator of covariant derivative D can be extended to the tensor algebra of (F, q, M) taking $D_\sigma f = \rho(\sigma)f$, assuming that it commutes with the contractions and behaves like a derivation with respect to tensor product. It comes out that if ω is a section in the dual (F^*, q^*, M) then

$$(D_s\omega)(\sigma) = \rho(s)\omega(\sigma) - \omega(D_s\sigma), \quad s \in \Gamma(E), \quad \sigma \in \Gamma(F)$$

and if a is a section in $L^2(F, \mathbb{R})$, then

$$(D_s a)(\sigma_1, \sigma_2) = \rho(s)a(\sigma_1, \sigma_2) - a(D_s\sigma_1, \sigma_2) - a(\sigma_1, D_s\sigma_2), \; s \in \Gamma(E), \; \sigma_1, \sigma_2 \in \Gamma(F). \tag{4.1}$$

Definition 4.1. The Riemannian metric g is called *compatible* with the A-connection D if $D_s g = 0$ for every $s \in \Gamma(E)$.

By (4.1) the condition of compatibility between g and D is equivalent to

$$\rho(s)g(\sigma_1, \sigma_2) = g(D_s\sigma_1, \sigma_2) + g(\sigma_1, D_s\sigma_2), \quad s \in \Gamma(E), \quad \sigma_1, \sigma_2 \in \Gamma(F). \tag{4.2}$$

Locally, (4.2) is written as follows

$$\rho_a^i(x)\frac{\partial g_{\alpha\beta}}{\partial x^i} = \Gamma_{\alpha a}^{\gamma}(x)g_{\gamma\beta}(x) + \Gamma_{\beta a}^{\gamma}(x)g_{\alpha\gamma}(x).$$

The operator D^α can be extended to α-section in the tensor bundles constructed with (F, q, M) and one deduces that

$$(D^\alpha g(t)(\sigma_1(t), \sigma_2(t)) = \left(\frac{g_{\mu\nu}}{dt} - g_{\mu\eta}\Gamma_{\mu a}^{\eta}y^a - g_{\eta\nu}\Gamma_{\mu a}^{\eta}y^a\right)\sigma_1(t)\sigma_2(t). \tag{4.3}$$

If (F, q, M) coincides with (E, p, M) we have

Theorem 4.1. *There exists an unique linear connection ∇ in the Lie algebroid A such that*

(i) $\nabla_s g = 0$,

(ii) $\nabla_{s_1} s_2 - \nabla_{s_2} s_1 = [s_1, s_2],\ s, s_1, s_2 \in \Gamma(E)$.

It is given by the formula

$$\begin{aligned} 2g\left(\nabla_{s_1} s_2, s_3\right) =& \rho(s_1) g(s_2, s_3) + \rho(s_2) g(s_1, s_3) - \rho(s_3) g(s_1, s_2) \\ & + g([s_3, s_1], s_2) + g([s_3, s_2], s_1) + g([s_1, s_2], s_3) \end{aligned} \tag{4.4}$$

and its local coefficients are given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\rho^i_b \frac{\partial g_{cd}}{\partial x^i} + \rho^i_c \frac{\partial g_{bd}}{\partial x^i} - \rho^i_d \frac{\partial g_{bc}}{\partial x^i} + L^e_{dc} g_{eb} + L^e_{db} g_{ec} - L^e_{bc} g_{ed} \right). \tag{4.5}$$

Proof. In the condition (i) written for $s_1, s_2, s_3 \in \Gamma(E)$ we cyclically permute s_1, s_2, s_3 and so we obtain two new identities. We add these and from the result we subtract the first. Using (ii) some terms cancel each other and we get (4.4). Writing (4.4) in a local basis of sections we find (4.5). The uniqueness follows by contradiction. □

If we put

$$T_\nabla(s_1, s_2) = \nabla_{s_1} s_2 - \nabla_{s_2} s_1 - [s_1, s_2], \quad s_1, s_2 \in \Gamma(E)$$

we get a section in the bundle $L(E, E; E)$ that may be called the torsion of ∇.

The curvature of ∇ is defined by

$$R_\nabla(s_1, s_2) s_3 = \nabla_{s_1} \nabla_{s_2} s_3 - \nabla_{s_2} \nabla_{s_1} s_3 - \nabla_{[s_1, s_2]} s_3, \quad s_1, s_2, s_3 \in \Gamma(E).$$

The connection ∇ given by the Theorem 4.1 is called the Levi-Civita connection of A.

We stress that the Theorem 4.1 says that given g there exists and is unique ∇ such that $\nabla g = 0$ and $T_\nabla = 0$. Now we give a different proof of this theorem.

Given g we may associate to it the energy function $\mathcal{E}\colon E \longrightarrow \mathbb{R}$, $\mathcal{E}(s) = g(s, s)$, $s \in E$. Locally, $\mathcal{E}(x, y) = g_{ab}(x) y^a y^b$, $s = y^a s_a$.

The energy function $\mathcal{E}$ is a regular Lagrangian on E i.e.

$$\det\left(\frac{1}{2} \frac{\partial^2 \mathcal{E}}{\partial y^a \partial y^b}\right) = \det(g_{ab}(x)) \neq 0.$$

In [2], we associated to any regular Lagrangian L on a Lie algebroid a semispray on E, that is a vector field

$$S = \rho^i_a y^a \frac{\partial}{\partial x^i} - 2G^a_L(x, y) \frac{\partial}{\partial y^a}$$

with

$$G^a_L = \frac{1}{4} g^{ab} \left(\frac{\partial^2 L}{\partial y^b \partial x^i} \rho^i_c y^c - \rho^i_b \frac{\partial L}{\partial x^i} - L^c_{bd} y^d \frac{\partial L}{\partial y^c} \right), \tag{4.6}$$

where $g_{ab} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^a \partial y^b}$ and (g^{ab}) is the inverse of the matrix (g_{ab}).

Taking $L = \mathcal{E}$ in (4.6), a direct calculation in which $L^a_{cd} y^c y^d = 0$ is used, shows that the semispray associated to $\mathcal{E}$ has the form

$$S = \rho^i_a y^a \frac{\partial}{\partial x^i} - \Gamma^a_{cd}(x) y^c y^d \frac{\partial}{\partial y^a}, \tag{4.7}$$

with Γ^a_{cd} given by (4.5). These coefficients determines ∇. They are symmetric in bottom indices, hence $T_\nabla = 0$. The uniqueness of ∇ follows by contradiction.

Note that (4.7) gives a 2-homogeneous semispray, that is a spray.

By (4.7) it follows

Theorem 4.2. *The integral curves of S are just the geodesics of the Levi-Civita connection ∇ in the Lie algebroid A.*

For a different derivation of S from g we refer to [24].

Now we come back to the general framework and prove

Lemma 4.1. *Let be the vector bundle (F, q, M) endowed with an A-connection D and a Riemannian metric g. Then for any A-path $\alpha \colon t \longrightarrow \alpha(t)$, $t \in [0, 1]$ on E with base curve $\gamma = q \circ \alpha = p \circ \alpha$ we have*

$$(D_\alpha g)(\sigma_1, \sigma_2) = \lim_{t \to 0} \frac{1}{t} \left[g_{\gamma(t)} \left(P^t_\alpha \sigma_1, P^t_\alpha \sigma_2 \right) - g_{\gamma(0)}(\sigma_1, \sigma_2) \right], \tag{4.8}$$

where $\sigma_1, \sigma_2 \in F_{\gamma(0)}$ and $P^t_\alpha \colon F_{\gamma(t)} \longrightarrow F_{\gamma(0)}$ is the parallel displacement defined by D along α.

Proof. Let $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ be the parallel $\alpha-$sections in (F, q, M) such that $\widetilde{\sigma_1}(0) = \sigma_1$, $\widetilde{\sigma_2}(0) = \sigma_2$. Then $P^t_\alpha \sigma_1 = \widetilde{\sigma_1}(t)$ and $P^t_\alpha \sigma_2 = \widetilde{\sigma_2}(t)$. By Taylor formula in the local basis (σ_β) we get $\left(P^t_\alpha \sigma_1\right)^\mu = \sigma^\mu_1 - \Gamma^\mu_{\nu a}(\tau) \widetilde{\sigma_1}^\nu(\tau) y^a(\tau) t$ for $\tau \in (0, t)$ and a similar formula for $P^t_\alpha \sigma_2$. Recall that $\alpha(t) = y^a(t) s_a$. Then by using again the Taylor formula and omitting the terms which contain t^2 and t^3, we may write

$$\begin{aligned} g_{\mu\nu}(\gamma(t)) \left(P^t_\alpha \sigma_1\right)^\mu \left(P^t_\alpha \sigma_2\right)^\nu - g_{\mu\nu}(\gamma(0)) \sigma^\mu_1 \sigma^\nu_2 = \\ = \left(g_{\mu\nu}(\gamma(0)) + \frac{dg_{\mu\nu}}{dt}(\gamma(0)) t \right) \left(P^t_\alpha \sigma_1\right)^\mu \left(P^t_\alpha \sigma_2\right)^\nu - g_{\mu\nu}(\gamma(0)) \sigma^\mu_1 \sigma^\nu_2 \\ = \left(\frac{dg_{\mu\nu}}{dt} - g_{\mu\eta} \Gamma^\eta_{\nu a} - g_{\eta\nu} \Gamma^\eta_{\mu a} \right) \sigma^\mu_1 \sigma^\nu_2 t, \end{aligned}$$

where the terms in the parenthesis are computed for some values τ in $(0, t)$.

Now dividing by t, taking $t \to 0$ and looking at (4.3), we obtain (4.8). □

Definition 4.2. *An A-conection D in (F, q, M) is said to be metrizable if there exists a Riemannian metric in (F, q, M) such that $Dg = 0$.*

Based on Lemma 4.1 we obtain:

Theorem 4.3. *Any A-conection D is metrizable with respect to g if and only if all its parallel displacements are isometries with respect to g.*

In particular, the holonomy group $\Phi(x)$ is made up of isometries of $(F_x,\, g(x))$.

Using a local chart around x we may put $\Phi(x)$ in an $1\colon 1$ correspondence with a subgroup $\widetilde{\Phi}(x)$ of $GL(k,\mathbb{R})$, $k = \operatorname{rank} F$. A change of local chart moves that subgroup in a conjugate of it. We identify $\Phi(x)$ with this class of conjugate subgroups in $GL(k,\mathbb{R})$. With this identification, by Theorem 4.3, we get

Theorem 4.4. *A necessary condition for an A-connection to be metrizable is that the holonomy group $\Phi(x)$ be a subgroup of the orthogonal group $O(k)$ for every $x \in M$.*

Indeed, if $\Phi(x)$ is made up of isometries of $(F_x, g(x))$, the elements of $\widetilde{\Phi}(x)$ are isometries of $(\mathbb{R}^k, <,>)$, with the inner product $<,>$ induced by $g(x)$.

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