J. Adv. Math. Studies
Vol. **3**(2010), No. 1, 117-124
http://journal.fairpartners.ro

THE WEIERSTRASS REPRESENTATION FOR MINIMAL IMMERSIONS IN THE LIE GROUP $\mathbb{H}_3 \times \mathbb{S}^1$

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Invited paper to celebrate Professor Constantin Udrişte, on the occasion of his seventies

Abstract. We give a setting for constructing a Weierstrass representation formula for simply connected minimal surfaces in the Lie group $\mathbb{H}_3 \times \mathbb{S}^1$. We derive the Weierstrass representation for surfaces and establish the generating equations for minimal surfaces in the Lie group $\mathbb{H}_3 \times \mathbb{S}^1$.

1. INTRODUCTION

Surface theory has been intensively studied in mathematics and physics. The application of the theory to solitary wave phenomena in physics yields socalled "soliton geometry". An important branch is the Weierstrass representation of the surface in constant curvature space. The representation makes us study surfaces and their properties by means of analysis methods. A classical example of such an approach is given by the Weierstrass representation for the minimal surface in $\mathbb{R}^3$.

Surfaces and their dynamics are key ingredients in a number of phenomena in physics too. They are, for instance, surface waves, propagation of flame fronts, growth of crystals, deformation of membranes, dynamics of vortex sheets, many problems of hydrodynamics connected with motion of boundaries between region of differing densities and viscosities. Number of papers has been devoted to a study and application of the integrals over surfaces in gauge field theories, string theory, quantum gravity and statistical physics (see [12]).

Direct approaches to describe surfaces always have been of great interest.The classical Weierstrass formulae for minimal surfaces immersed in the three-dimensional Euclidean space $\mathbb{R}^3$ is the best known example of such an approach. Recently the Weierstrass formulae have been generalized to the case of generic surfaces in $\mathbb{R}^3$. During the last two years the generalized Weierstrass formulae have been used intensively to study both global properties of surfaces in $\mathbb{R}^3$ and their integrable deformations. Analytic methods to study surfaces and their properties are of great interest both

Received: July 15, 2009.
2000 *Mathematics Subject Classification*: 53C42, 53A10.
Key words and phrases: Weierstrass representation, minimal surface, Lie group.

in mathematics and in physics. A classical example of such an approach is given by the Weierstrass representation for minimal surfaces (see [3]). This representation allows us to construct any minimal surface in the three-dimensional Euclidean space $\mathbb{R}^3$ via two holomorphic functions. It is the most powerful tool for the analysis of minimal surfaces.

D. A. Berdinski and I. A. Taimanov gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators (see [2]).

It is well-known that the classical Weierstrass-Enneper representation formula describes minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ in terms of their Gauss maps and auxiliary holomorphic functions (see [13]). More generally, a remarkable representation formula has been discovered by Kenmotsu (see [3]) for arbitrary surfaces in $\mathbb{R}^3$ with nonvanishing mean curvature, which describes these surfaces in terms of their Gauss maps and mean curvature functions. On the other hand, Kobayashi (see [5, 6]) proved the Lorentzian version of the classical Weierstrass-Enneper representation formula for maximal surfaces in Minkowski 3-space $\mathbb{L}^3$ (see [8]) and applied it to the study of maximal surfaces with conelike singularities.

In this paper we describe a method to derive a Weierstrass-type representation formula for simply connected immersed minimal surfaces in the 4-dimensional Lie group $\mathbb{H}_3 \times \mathbb{S}^1$. Then, we shall give a Weierstrass-type representation formula for minimal surfaces in the 4-dimensional Lie group $\mathbb{H}_3 \times \mathbb{S}^1$.

2. PRELIMINARIES

Let (M, g) be an n-dimensional Riemannian manifold and $\Sigma \subset M$ be a Riemannian surface and $\zeta\colon \Sigma \to M$ a smooth map. The pull-back bundle $\zeta^*(TM)$ has a metric and compatible connection, the pull-back connection, induced by the Riemannian metric and the Levi-Civita connection of M. Consider the complexified bundle $\mathbb{E} = \zeta^*(TM) \otimes \mathbb{C}$.

Let (u, v) be local coordinates on Σ, and $z = u+iv$ the (local) complex parameter and set, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

The pull-back connection extends to a complex connection on $\mathbb{E}$, hermitian with respect to $\langle \cdot, \cdot \rangle$ and it is well known that $\mathbb{E}$ has a unique holomorphic structure such that a section $W\colon \Sigma \to \mathbb{E}$ is holomorphic if and only if:

$$\nabla_{\frac{\partial}{\partial \overline{z}}} W = 0,$$

where ∇ is the pull-back connection.

Let

$$\left.\frac{\partial \zeta}{\partial u}\right|_p = \zeta_{*p}\left(\left.\frac{\partial}{\partial u}\right|_p\right), \qquad \left.\frac{\partial \zeta}{\partial v}\right|_p = \zeta_{*p}\left(\left.\frac{\partial}{\partial v}\right|_p\right)$$

and

$$\phi = \zeta_z = \frac{\partial \zeta}{\partial z} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial u} - i\frac{\partial \zeta}{\partial v}\right).$$

Let now $\zeta\colon \Sigma \to M$ be a conformal immersion and $z = u + iv$ a local conformal parameter. Then the induced metric is

$$ds^2 = \lambda^2(du^2 + dv^2) = \lambda^2|dz|^2.$$

The Beltrami–Laplace operator on M, with respect to the induced metric is given by

$$\Delta = \lambda^{-2}\left(\frac{\partial}{\partial u}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{\partial}{\partial v}\right).$$

We recall that a map $\zeta\colon \Sigma \to M$ is harmonic if its tension field

$$\tau(\zeta) = \mathrm{trace}\nabla d\zeta = 0.$$

Let $\{x_1, \dots, x_n\}$ be a system of local coordinates in a neighborhood U of M such that $U \cap \zeta(\Sigma) \neq \emptyset$. Then, in an open set $\Omega \subset \Sigma$,

$$\phi = \sum_{j=1}^{n} \phi_j \frac{\partial}{\partial x_j}$$

for some complex-valued functions ϕ_j defined on Ω. With respect to the local decomposition of ϕ, the tension field can be written as:

$$\begin{aligned}
\tau(\zeta) &= \sum_i \left\{ \Delta\zeta_i + 4\lambda^{-2} \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{\partial \zeta_j}{\partial \overline{z}} \frac{\partial \zeta_k}{\partial z} \right\} \frac{\partial}{\partial x_i} \\
&= 4\lambda^{-2} \sum_i \left\{ \frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \overline{\phi}_j \phi_k \right\} \frac{\partial}{\partial x_i},
\end{aligned}$$

where Γ^i_{jk} are the Christoffel symbols of M.

The section ϕ is holomorphic if and only if

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial \overline{z}}} \left(\sum_{i=1}^{n} \phi_i \frac{\partial}{\partial x_i} \right) &= \sum_i \left\{ \frac{\partial \phi_i}{\partial \overline{z}} \frac{\partial}{\partial x_i} + \phi_i \nabla_{\frac{\partial \zeta}{\partial \overline{z}}} \frac{\partial}{\partial x_i} \right\} \\
&= \sum_i \left\{ \frac{\partial \phi_i}{\partial \overline{z}} \frac{\partial}{\partial x_i} + \phi_i \nabla_{\sum_j \overline{\phi}_j \frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \right\} \\
&= \sum_i \left\{ \frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k} \Gamma^i_{jk} \overline{\phi}_j \phi_k \right\} \frac{\partial}{\partial x_i} \\
&= 0,
\end{aligned}$$

thus if and only if

$$\frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k} \Gamma^i_{jk} \overline{\phi}_j \phi_k = 0, \quad i = 1, 2, \dots, n. \tag{2.1}$$

From (2.1) and the expression of the tension field we have that

$$4\lambda^{-2}\left(\nabla_{\frac{\partial}{\partial \bar{z}}}\phi\right) = \tau(\zeta).$$

Thus $\zeta\colon \Sigma \to M$ is harmonic if and only if ϕ is a holomorphic section of $\mathbb{E}$. Now, since a conformal map from a surface to a Riemannian manifold is harmonic if and only if it is a minimal immersion. We conclude that a conformal immersion is minimal if and only if ϕ is a holomorphic section of $\mathbb{E}$.

Theorem 2.1 ([13])**.** *Let (M, g) be a Riemannian manifold and $\{x_1, \ldots, x_n\}$ local coordinates. Let ϕ_j, $j = 1, \ldots, n$, be complex-valued functions in an open simply connected domain $\Omega \subset C$ which are solutions of* (2.1)*. Then the map*

$$\zeta_j(u, v) = 2\,\mathrm{Re}\left(\int_{z_0}^{z}\phi_j dz\right) \tag{2.2}$$

is well defined and defines a minimal conformal immersion if and only if the following conditions are satisfied:

$$\text{i)}\ \sum_{j,k=1}^{n} g_{ij}\phi_j\overline{\phi_k} \neq 0; \qquad \text{ii)}\ \sum_{j,k=1}^{n} g_{ij}\phi_j\phi_k = 0.$$

3. THE LIE GROUP $\mathbb{H}_3 \times \mathbb{S}^1$

Heisenberg group $\mathbb{H}_3$ is formed by all matrices of the form

$$\mathbb{H}_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

with the group multiplication induced by the standard matrix product. The Riemannian metric g given by

$$g = dx^2 + dy^2 + (dz + xdy)^2.$$

The Lie algebra of $\mathbb{H}_3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.$$

Now let C_{ij}^{k} be the structure's constants of the Lie algebra g of G (see [9]) that is,

$$[e_i, e_j] = C_{ij}^{k} e_k.$$

The corresponding Lie brackets are

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = [e_2, e_3] = 0.$$

The Kozsul formula for the Levi–Civita connection is:

$$2g(\nabla_{e_i} e_j, e_k) = C_{ij}^{k} - C_{jk}^{i} + C_{ki}^{j} := L_{ij}^{k}, \tag{3.1}$$

where the non zero L_{ij}^{k} 's are

$$L_{12}^{3} = 1, \quad L_{21}^{3} = -1, \quad L_{13}^{2} = -1, \quad L_{31}^{2} = -1, \quad L_{23}^{1} = 1, \quad L_{32}^{1} = 1.$$

$\mathbb{H}_3 \times \mathbb{S}^1$ is closed connected subgroup of $GL(4, \mathbb{C})$ defined by

$$\left\{ \begin{bmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi it} \end{bmatrix} : x, y, z, t \in \mathbb{R} \right\}.$$

Note that the left invariant metric can be written as:

$$ds^2 = \left(\omega^1\right)^2 + \left(\omega^2\right)^2 + \left(\omega^3\right)^2 + \left(\omega^4\right)^2,$$

where

$$\omega^1 = dx, \quad \omega^2 = dy, \quad \omega^3 = dz - xdy, \ \omega^4 = dt$$

is the left invariant orthonormal coframe associated with the orthonormal left-invariant frame

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \tag{3.2}$$

The corresponding Lie brackets are

$$[e_1, e_2] = e_3, \ \ [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0.$$

With respect to this orthonormal basis, the Levi-Civita connection can be easily computed as:

$$\begin{aligned} &\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = 0, \quad \nabla_{e_1} e_2 = e_3, \\ &\nabla_{e_1} e_3 = -\ \nabla_{e_3} e_1 = \ \nabla_{e_1} e_4 = \nabla_{e_4} e_1 = -\frac{1}{2} e_2, \\ &\nabla_{e_2} e_3 = -\ \nabla_{e_3} e_2 = \ \nabla_{e_2} e_4 = -\ \nabla_{e_4} e_2 = \frac{1}{2} e_1. \end{aligned} \tag{3.3}$$

From (3.1) and (3.3) we have

$$\begin{aligned} &L^3_{12} = 1, \ L^1_{23} = \frac{1}{2}, \ L^1_{32} = -\frac{1}{2}, \ L^1_{24} = \frac{1}{2}, \ L^1_{42} = -\frac{1}{2}, \\ &L^2_{13} = -\frac{1}{2}, \ L^2_{31} = \frac{1}{2}, \ L^2_{14} = -\frac{1}{2}, \ L^2_{41} = \frac{1}{2}. \end{aligned} \tag{3.4}$$

4. MINIMAL SURFACES IN THE LIE GROUP $\mathbb{H}_3 \times \mathbb{S}^1$

Let us expand Ψ with respect to this basis to obtain

$$\Psi = \sum_{k=1}^{4} \psi_k e_k.$$

Since the parameter z is conformal, we have

$$\langle \Psi, \Psi \rangle = 0,$$

which is rewritten as

$$\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 = 0. \tag{4.1}$$

The equation (4.1) implies that the vector Ψ can be parametrized in the form

$$\psi_1 = \frac{i}{2}(\varphi_1 + \varphi_2),\ \psi_2 = \frac{1}{2}(\varphi_1 - \varphi_2), \qquad (4.2)$$
$$\psi_3 = \frac{1}{2}(\varphi_1 + \varphi_2),\ \psi_4 = \frac{i}{2}(\varphi_1 - \varphi_2).$$

Setting

$$\phi = \sum_{i=1}^{4} \phi_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{4} \psi_i e_i \qquad (4.3)$$

for some complex functions $\phi_i, \psi_i \colon \Omega \subset \mathbb{C}$. Moreover, there exists an invertible matrix $A = (A_{ij})$, with function entries $A_{ij} \colon \zeta(\Omega) \cap U \to \mathbb{R}, i, j = 1, 2, 3, 4$, such that

$$\phi_i = \sum_{j=1}^{4} A_{ij} \psi_j.$$

The Kozsul formula for the Levi–Civita connection is:

$$2g(\nabla_{e_i} e_j, e_k) = L_{ij}^k.$$

Using the expression of ϕ, the section ϕ is holomorphic if and only if

$$\frac{\partial \psi_i}{\partial \overline{z}} + \frac{1}{2} \sum_{j,k} L_{jk}^i \overline{\psi}_j \psi_k = 0, \quad i = 1, 2, 3, 4. \qquad (4.4)$$

Theorem 4.1. *Let ψ_j, $1 \leq j \leq 4$ be complex-valued functions defined in a open simply connected set $\Omega \subset C$, such that the following conditions are satisfied:*

i) $|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \neq 0$*;*

ii) $\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 = 0$*;*

iii) ψ_j *are solutions of* (4.4).

Then, the map $\zeta \colon \Omega \to H_3 \times S^1$ defined by

$$\zeta_i(u, v) = 2\,\mathrm{Re}\left(\int_{z_0}^{z} \sum_j A_{ij} \psi_j dz \right) \qquad (4.5)$$

is a conformal minimal immersion.

Proof. By Theorem (2.1) we see that ζ is a harmonic map if and only if ζ satisfy (4.5). Then, the map ζ is a conformal minimal immersion. □

Lemma 4.1. *If* Ψ *satisfies the equation* (4.4) *then*

$$\begin{aligned}
\frac{\partial\psi_1}{\partial\overline{z}} + \frac{1}{8}(\overline{\varphi}_1\varphi_2 - \overline{\varphi}_2\varphi_1) - \frac{i}{8}(|\varphi_1|^2 - \overline{\varphi}_1\varphi_2 - \overline{\varphi}_2\varphi_1 + |\varphi_1|^2) = 0,\\
\frac{\partial\psi_2}{\partial\overline{z}} - \frac{1}{8}(\overline{\varphi}_1\varphi_2 - \overline{\varphi}_2\varphi_1) - \frac{i}{8}(|\varphi_1|^2 + \overline{\varphi}_1\varphi_2 + \overline{\varphi}_2\varphi_1 + |\varphi_1|^2) = 0,\\
\frac{\partial\psi_3}{\partial\overline{z}} - \frac{i}{16}(|\varphi_1|^2 - \overline{\varphi}_1\varphi_2 + \overline{\varphi}_2\varphi_1 - |\varphi_1|^2) = 0,\\
\frac{\partial\psi_4}{\partial\overline{z}} = 0.
\end{aligned} \tag{4.6}$$

Proof. Substitute (3.4), and (4.2) into (4.4) we have (4.6). □

Lemma 4.2. *If* Ψ *satisfies the equation* (4.4) *then*

$$\frac{\partial\varphi_1}{\partial\overline{z}} = -\frac{1}{4}(\overline{\varphi}_1\varphi_2 + \overline{\varphi}_2\varphi_1). \tag{4.7}$$

Proof. Using the forth equation of (4.6) we have

$$\frac{\partial\varphi_1}{\partial\overline{z}} = \frac{\partial\varphi_2}{\partial\overline{z}}. \tag{4.8}$$

The pair consisting of the first and second equations of (4.6) is equivalent to:

$$\frac{\partial\psi_1}{\partial\overline{z}} + \frac{\partial\psi_2}{\partial\overline{z}} + \frac{i}{4}(\overline{\varphi}_1\varphi_2 + \overline{\varphi}_2\varphi_1).$$

From (4.8) we obtain (4.7). □

Theorem 4.2. *Let* φ_1 *and* φ_2 *be complex-valued functions defined in a simply connected domain* $\Omega \subset C$*. Then the map* $\zeta\colon \Omega \to H_3 \times S^1$*, defined by*

$$\begin{aligned}
\zeta_1(u,v) &= \operatorname{Re}\left(i\int_{z_0}^{z}(\varphi_1+\varphi_2)\,dz\right),\\
\zeta_2(u,v) &= \operatorname{Re}\left(\int_{z_0}^{z}(\varphi_1-\varphi_2)\,dz\right),\\
\zeta_3(u,v) &= \operatorname{Re}\left(\int_{z_0}^{z}((\varphi_1-\varphi_2)\,\zeta_1+(\varphi_1+\varphi_2))dz\right),\\
\zeta_4(u,v) &= \operatorname{Re}\left(i\int_{z_0}^{z}(\varphi_1-\varphi_2)\,dz\right),
\end{aligned} \tag{4.9}$$

is a conformal minimal immersion.

Proof. By substituting (3.2) in (4.3) we get

$$\phi_1 = \psi_1,\ \phi_2 = \psi_2,\ \phi_3 = \psi_3 + x\psi_2,\ \phi_4 = \psi_4.$$

From (2.2) we have (4.9). Using Theorem 4.1 $\zeta\colon \Omega \to \mathbb{H}_3 \times \mathbb{S}^1$ is a conformal minimal immersion. □

For other developments of Weierstrass representation for minimal immersions in the Lie groups , see [1], [4], [7], [11]÷ [14] and [9]÷ [19].

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