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DESCRIBING PARAMETRIC FAMILIES OF ALGEBRAIC POINTS OF DEGREE AT MOST ℓ ON THE SUPERELLIPTIC CURVE FAMILY $\mathcal{C}_b$ DEFINED BY $y^2 = x^5 + b^5$

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Abstract. We give an explicit description of the set of algebraic points of arbitrary degree on the family of hyperelliptic curves $\mathcal{C}_b$, studied in [19] by Jędrzejak and defined by the affine equation $y^2 = x^5 + b^5$, where $b \notin \mathbb{Q}^{\times 2}$. This family is a special case of the superelliptic family of curves $\mathcal{C}_{q,p,a}$ defined by $y^q = x^p + a$, with q and p prime numbers and a an integer. This more general family was also studied in [20] by Jędrzejak, who, using height theory, explicitly determined the set $\mathcal{C}_b^{(1)}(\mathbb{Q})$ of rational points (see Theorem 1.1). The purpose of this note is to extend those results to algebraic points of arbitrary degree. We first construct an explicit $\mathbb{Q}$-basis of the vector spaces $\mathcal{L}(\ell\infty)$, for $\ell \in \mathbb{N}$, and we provide an explicit description of the Mordell-Weil group of rational points on the Jacobian $\mathcal{J}_b(\mathbb{Q})$. We then apply a fundamental form of the Abel-Jacobi theorem [14, 16] to characterize principal divisors associated with rational functions under consideration. Finally, by introducing a parameter γ, we explicitly construct a union of families of algebraic points, denoted $\mathcal{C}_b^{(\ell)}(\mathbb{Q})$.

REFERENCES

[1] P. Alvanos: *Riemann-Roch Spaces and Computation*, de Gruyter, Warsaw, Poland, 2015.
[2] T. Andreescu, D. Andrica and I. Cucurezeanu: *An Introduction to Diophantine Equations: A Problem-Based Approach*, Birkhäuser, 2010.
[3] D. Abramovich and J. Harris: *Abelian varieties and curves in* $\mathcal{W}_d(\mathbb{C})$, Compos. Math., **78**(1991), No. 2, 227-238.
[4] I.G. Bashmakova: *Diophantus and Diophantine equations*, American Mathematical Society, 2019 (DME 54).
[5] E. Bombieri and W. Gubler: *Heights in Diophantine Geometry*, Cambridge University Press, 2009 (NMM 4).
[6] M. Coppens and G. Martens: *Secant spaces and Clifford's theorem*, Compos. Math., **78**(1991), No. 2, 193-212.
[7] I. Déchène: Arithmetic of generalized Jacobians, in Algorithmic Number Theory (Florian Hess, Sebastian Pauli, Michael Pohst (Eds)), 2006 (LNCS 4076), pp. 421-435.
[8] M.M.D. Diallo: *Analytic parametrization of the algebraic points of given degree on the curve of affine equation* $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$, Transylvanian J. Math. Mechanics, **15**(2023), 41-50.
[9] M.M.D. Diallo: *Explicit family of algebraic points of given degree of the curve family* $\mathcal{C}_q$, J. Adv. Math. Stud., **17**(2024), No. 4, 444-450.
[10] M.M.D. Diallo and C.M. Coly: *Generalisation of algebraic point families of the Fermat curve quotient* $\mathcal{C}_{1,2}(7)$, J. Contemp. Appl. Math., **15**(2025), No. 1, 107-121.

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[11] M. Fall: *Parametrization of algebraic points of low degrees on the Schaeffer curve*, J. Math. Sci. Modelling, **4**(2021), No. 2, 51-55.
[12] G. Faltings: Finiteness theorems for abelian varieties over number fields: Arithmetic geometry, in *Arithmetic Geometry* (G. Cornell *et al.* (Eds)), Springer-Verlag New York Inc., 1986, pp. 9-26.
[13] G. Frey: *The way to the proof of Fermat's last theorem*, Ann. Fac. Sci. Toulouse Math. (6), **18**(2009), No. 2, 5-23.
[14] S. Gmira: *Abel-Jacobi theorem*, 19 Jul 2015, arXiv:1507.05345v1 [math.GM], 11 pages.
[15] J.J. Gray: *The Riemann-Roch theorem and geometry, 1854-1914*, Doc. Math. (Proceedings of the International Congress of Mathematicians), **III**(1998), 811-822.
[16] P.A. Griffiths: *Introduction to Algebraic Curves*, American Mathematical Society, Providence, 1989 (TMM 76).
[17] J. Harris: *Algebraic Geometry. A First Course*, Springer Science & Business Media, 1992 (GTM 133).
[18] M. Hindry and J. H. Silverman: The geometry of curves and abelian varieties, in *Diophantine Geometry. An Introduction* (Marc Hindry and Joseph H. Silverman (Eds)), Springer, 2000 (GTM 201), pp. 6-167.
[19] T. Jędrzejak: *Ranks of quadratic twists of Jacobians of generalized Mordell curves*, J. Algebra, **663**(2025), 565-588.
[20] T. Jędrzejak: *A note on the torsion of the jacobians of superelliptic curves* $y^q = x^p + a$, Banach Center Publ., **108**(2016), No. 1, 143-149.
[21] T. Jędrzejak: *On the torsion of the jacobians of superelliptic curves* $y^q = x^p + a$, J. Number Theory, **145**(2014), 402-425.
[22] A. Mattuck and A. Mayer: *The Riemann-Roch theorem for algebraic curves*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **17**(1963), No. 3, 223-237.
[23] L.J. Mordell: *Diophantine Equations*, Academic Press, 1969 (PAM 30).
[24] B. Poonen: *Rational Points on Varieties*, American Mathematical Society, 2017 (GSM 186).
[25] J.H. Silverman and J.T. Tate: *Rational Points on Elliptic Curves*, Springer, 1992 (UTM 9).
[26] O. Sall: *Points algébriques sur certains quotients de courbes de Fermat*, C. R. Math. Acad. Sci. Paris, **336**(2003), No. 2, 117-120.
[27] P. Vojta: *Siegel's theorem in the compact case*, Ann. of Math. (2), **133**(1991), No. 3, 509-548.
[28] M.E. Walker: *The morphic Abel-Jacobi map*, Compos. Math., **143**(2007), No. 4, 909-944.

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