

HARMONIC-LIKE BIFUNCTION EQUILIBRIUM PROBLEMS

MUHAMMAD ASLAM NOOR AND KHALIDA INAYAT NOOR

ABSTRACT. In this paper, some new classes of harmonic-like bifunction equilibrium problems are introduced and investigated. The auxiliary principle technique is used to consider some new predictor-corrector and inertial proximal algorithms for solving general equilibrium problems. The convergence of the proposed methods either requires partially relaxed strongly monotonicity or pseudomonotonicity, which are weaker conditions. As special cases, we obtain a number of known and new results for solving various classes of equilibrium and variational inequality problems. Our results represent can be viewed as significant improvement and refinement of the previously known results.

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University of Wah
Department of Mathematics
Wah Cantt, Pakistan
Email address: noormaslam@gmail.com

University of Wah
Department of Mathematics
Wah Cantt, Pakistan
Email address: khalidan@gmail.com